COUNTY COUNCIL OF THE COUNTY OF LANARK
EDUCATION COMMITTEE

.... School.

(1) This book is the property of the Education Authority, and is intended for the use of scholars attending Schools in the County of Lanark.

(2) Parents and Guardians are asked to co-operate with the Teachers in seeing that the Book is kept clean and in good repair.

(3) When a leaf becomes loose or gets torn, it should at once be shown to the Teacher of the class

(4) The Parent or Guardian of a scholar losing or misusing a book will be required to replace it, or to refund to the Education Authority the value of the book

(5) The book must be covered by the pupil to whom it is issued.

Date	Name	Class
7/4/61	Gillian Ross	IV.C.
'62	Elizabeth Adam	IV.C.
3.-64	LINDA [illegible]	[illegible]
1.3.4	Ian Bonalie	3M.3
19[illegible]	[illegible]	1A

A NEW COURSE IN ALGEBRA

By the same authors

A NEW COURSE IN GEOMETRY

Complete with Answers

Part I without Answers

Part II without Answers

A NEW COURSE IN ARITHMETIC

Complete with Answers

Complete without Answers

Part I without Answers

Part II without Answers

A NEW COURSE IN ALGEBRA

by

A. WALKER, M.A., B.Sc.
Formerly Headmaster, Jordanhill College School, Glasgow

and

J. MILLAR, M.A. (Hons).
Principal Teacher of Mathematics, Dollar Academy

PART I

WITHOUT ANSWERS

1724

LONGMANS

Printed in Great Britain by Richard Clay and Company, Ltd.,
Bungay, Suffolk

PREFACE

THIS book is intended to provide a five-year course in Algebra which will lay a sound foundation for later, more specialised study of the subject.

At the beginning, the usual operations in Arithmetic are generalised with the use of letters, only positive numbers being used. Negative numbers, however, are introduced fairly early, as this procedure is widely followed.

The exercises throughout are numerous and graded.

A certain sequence had to be adopted, and certain methods used in the worked examples, and several topics, e.g. Factors, Fractions, Ratio and Proportion, Variation, Indices, are dealt with more than once at stages of increasing difficulty, but it is hoped that teachers will be able to use the book with their own sequence and methods. The detailed explanations of bookwork are included to enable the student to work as far as possible without frequent reference to the teacher. In the treatment of graphs, emphasis is laid on interpretation and adaptation.

Part I deals with the usual topics, including equations and problems as far as quadratic equations and changing the subject of a formula, and there are introductions to Quadratic Surds, Ratio and Proportion, Variation, and Indices. Statistical graphs, the straight-line graph and graphs of quadratic and cubic functions and the function m/x are included.

Part II starts with the Remainder Theorem and deals with Indices, Logarithms, Surds, Ratio and Proportion, Variation, Theory of Quadratics, etc., up to and including the Simple Series. There is a chapter dealing with Compound Interest and Annuities and one dealing with graphs of more complex

functions. Logarithms are closely linked with indices, and particular attention has been paid to certain topics which seem to cause difficulty, including Variation, the Quadratic function and the sum to Infinity of the Geometric Series.

There are fifty-five revision papers providing just over three hundred examples, and one hundred questions (taken from the usual school examinations) in two sets of examination papers, one set at the end of each part.

The book covers the work in Algebra for the Scottish Leaving Certificates, (Ordinary and Higher Grades) and for the English Certificate of Education (Ordinary Level).

We are indebted to the following examining bodies for permission to reproduce examination questions; and the block capital at the end of each question indicates the source of the question: [S] The Controller of Her Majesty's Stationery Office (for Scottish Leaving Certificate questions); [P] The Scottish Universities Entrance Board; [C] Local Examinations Syndicate, University of Cambridge; [Q] Local Examinations Syndicate, University of Oxford; [L] The Senate of London University; [N] Joint Matriculation Board of the Northern Universities; [B] Southern Universities Joint Board for School Examinations; [W] Welsh Joint Education Committee; [D] Examination Board of the University of Durham.

CONTENTS

Part I

CHAPTER		PAGE
1.	The Use of Letters	1
2.	Formulæ and Substitution	8
3.	Easy Equations and Problems	12
4.	The Four Rules	15
	Revision Papers 1–5	21
5.	Directed Numbers	25
6.	Brackets I	39
7.	The Four Rules with Directed Numbers	44
8.	Substitution with Directed Numbers	51
	Revision Papers 6–10	55
9.	H.C.F.—L.C.M.—Fractions I	59
10.	Brackets II	71
11.	Simple Equations	74
	Revision Papers 11–15	84
12.	Symbolical Expression	88
13.	Problems involving Simple Equations	96
14.	Graphs I—Statistical	106
	Revision Papers 16–20	125
15.	Simultaneous Equations of the First Degree with Two Unknowns	129
16	Problems involving Simultaneous Equations of the First Degree in Two Unknowns	138
17.	Multiplication and Division by Binomials and Trinomials	146
	Revision Papers 21–25	163
18.	Factors	167
19.	Fractions II	182
20.	Quadratic Equations and Problems I—Solution by Factors	193

CHAPTER		PAGE
21.	Graphs II—The Straight Line	199
	Revision Papers 26–30	216
22.	Quadratic Equations and Problems II—Completing the Square—Quadratic Surds	221
23.	Literal Equations—Changing the Subject of a Formula	237
24.	Ratio and Proportion I	244
25.	Variation I	254
26.	Indices I	272
27.	Graphs III—Quadratic and Cubic Functions—the Function m/x	279
	Revision Papers 31–35	298
	Examination Papers 1–10	302
	Table of Square Roots	312

PART I

CHAPTER I

THE USE OF LETTERS

IN Arithmetic we know that, when any two numbers are linked by the plus sign, they are to be added, e.g.

$2\frac{1}{2} + 5$ means that 5 is to be added to $2\frac{1}{2}$.

Similarly,

$6 - 3\frac{1}{4}$ means that $3\frac{1}{4}$ is to be subtracted from 6.
$4\frac{2}{3} \times 3$ means that $4\frac{2}{3}$ is to be multiplied by 3.
$9 \div 2$ or $\frac{9}{2}$ means that 9 is to be divided by 2.

In Algebra we use letters to represent numbers, and the signs $+$, $-$, $\times$, $\div$ are used with the same meanings as in Arithmetic, e.g.

$a + b$ means that b is to be added to a.
$d - c$ means that c is to be subtracted from d.
$x \times y$ means that x is to be multiplied by y.
$b \div c$ or $\dfrac{b}{c}$ means that b is to be divided by c.

It should be noted carefully that:

1. A letter can represent any number, e.g. the letter a could represent 15 or $2\frac{2}{3}$ or $\frac{3}{7}$.

2. $x \times y$ may be written $x \cdot y$, or more usually xy. When 4 and a are multiplied, the result might be written $4 \times a$ or $4a$. It might also be written as $a \times 4$, but it is never written $a4$.

3. In Arithmetic 43 means forty-three or $4 \times 10 + 3$, whereas in Algebra ab means $a \times b$.

4. In Algebra, as in Arithmetic, numbers within brackets are looked on as single numbers, e.g.

$$3(4 + 7) = 3(11) = 33$$

and $a(b + c)$ means that the sum of b and c is to be multiplied by a.

Write down the following, using only letters and signs:

Example 1: Add a and b, and subtract the result from c.

$a + b$ is the sum of a and b. This is to be looked on as a single number, so we write it in the form $(a + b)$.

To subtract this number from c we write $c - (a + b)$.

Example 2: Multiply the sum of a and b by 3, and divide the result by x.

The sum of a and b, written as a single number, is $(a + b)$.

Multiplying this number by 3 we get $3(a + b)$.

Dividing this result by x we get $3(a + b) \div x$ or $\dfrac{3(a + b)}{x}$.

Exercises 1

Write down, without simplifying, using only letters and signs:

1. 3 added to 4, 3 added to a, c added to a.

2. 4 subtracted from 7, a subtracted from 7, a subtracted from b.

3. 4 multiplied by 3, x multiplied by 3, x multiplied by y.

4. a divided by 5, 5 divided by a, a divided by b.

5. The number which is 2 more than x.

6. The number which is 3 less than a.

7. The number which is c more than d.

8. The number which is a less than b.

9. The number which is 3 times x.

10. The number which is one-half of a.

11. The number which is twice the sum of a and b.

12. The number which is twice the product of a and b.

13. Add a and b, and subtract the result from x.

14. Subtract a from b, and multiply the result by 3.

15. Multiply x by 4, and add y to the result.

16. Multiply a by 3. Take b from the answer. Divide the resulting number by 4.

17. From the sum of x and y subtract the sum of a and b.

18. Find two-thirds of the difference between x and y, if y is greater than x.

19. Subtract twice x from y. Multiply the answer by a. Divide the resulting number by the product of b and c.

20. Add a and b. Multiply the answer by the sum of c and d. Divide the result by k.

Powers

Since $2 \times 3 = 6$, 2 and 3 are factors of the product 6. Similarly, 2, 3 and 5 are factors of the product 30.

$$3 \times 3 \text{ is the product of the factors 3 and 3.}$$

Here both factors are the same, and hence the product is called a *power*.

Since there are 2 factors, 3×3 is the second power of 3, written 3^2. Similarly, $3 \times 3 \times 3$ is the third power of 3, written 3^3, and $a \times a \times a \times a$ is the fourth power of a, written a^4.

The small figures 2, 3, 4, which show the number of factors in the power, are called *indices*. Each is an *index*.

3×3 is said to be the *square* of 3.

If we consider a square with each side 3 units of length, its area would be 3×3 or 3^2 (read 3 squared) units of area.

$3 \times 3 \times 3$ is said to be the cube of 3.

If we consider a cube with each edge 3 units of length, its volume would be $3 \times 3 \times 3$ or 3^3 (read 3 cubed) units of volume.

Note carefully the difference between:

1. $2a$, which means two times a, or a taken two times, i.e. $a + a$, and a^2, which means a multiplied by a, or $a \times a$.

2. $3a$, which means three times a, or a taken three times, i.e. $a + a + a$, and a^3, which means a multiplied by a multiplied by a, i.e. $a \times a \times a$.

3. $2a^3 = 2 \times a \times a \times a$, and $(2a)^3 = 2a \times 2a \times 2a$.

Roots

Since $3 \times 3 = 3^2$, we say that the square root of 3^2 is 3.

This is written $\sqrt{3^2} = 3$

Similarly, $\sqrt{a^2} = a$

Since $5 \times 5 \times 5 = 5^3$, we say that the cube root of 5^3 is 5.

This is written $\sqrt[3]{5^3} = 5$

Similarly, $\sqrt[3]{a^3} = a$

Example 1: Write down the square of the sum of a and b.

The sum of a and b, written as a single number, is $(a + b)$.

$\therefore$ The square of this number is $(a + b)^2$.

Example 2: Write down the sum of the squares of a and b.

The square of a is a^2

The square of b is b^2.

$\therefore$ The sum of these squares is $a^2 + b^2$.

Exercises 2

Write down the following:

1. Square b and add 3 to the result.
2. Subtract the square of a from 7.
3. Add the square of a to the cube of b.
4. Subtract b from c and square the result.
5. The cube of the sum of a and b.
6. The sum of the cubes of a and b.

7. Multiply the square of a by the square of b.

8. The sum of the square of a and the square root of a.

9. The square root of the sum of a and b.

10. If b is greater than a, write down the cube of their difference.

Write down, as shortly as possible:

11. $a + a + a$

12. $a + a + a + a + a$

13. $a \times a \times a \times a$

14. $x \times x \times x \times x \times x \times x$

15. $2 \times a \times a$

16. $5 \times b \times b \times b$

17. $aabb$

18. $\dfrac{xxx}{aaa}$

19. $\dfrac{x + x + x}{a + a}$

20. $\dfrac{x \times x \times x}{a + a + a}$

Write down in fully expanded form:

21. a^4

22. $3a^2$

23. $(3a)^2$

24. a^2b

25. ab^2

26. $2a^2b^3$

27. $a^2 + 2a$

28. $a^2 - b^2$

29. $(2m)^4$

30. $2m^4$

In Arithmetic you have worked exercises in the usual weights and measures, and it is useful practice to work a few examples, using letters instead of numbers.

Example 1: Express £c in shillings.

To express £3 in shillings we multiply 3 by 20.

$\therefore$,, £c ,, ,, ,, c by 20.

$\therefore$ £$c = 20c$ shillings.

Example 2: A motor car runs 40 miles to the gallon of petrol. How many gallons are required for a journey of k miles?

Suppose we put the number 90 in place of k.

A journey of 40 miles requires 1 gal.

,, 90 ,, $\frac{90}{40}$ gal.

$\therefore$ A journey of k miles requires $\frac{k}{40}$ gal. (replacing the number 90 by k).

Example 3: A square field has the length of each side x ft How many yards is it right round the field?

The square field has 4 sides, each of which is x ft. long.

$$\therefore \text{ Total distance round the field} = 4x \text{ ft.}$$

To express $4x$ ft. in yd. we divide $4x$ by 3.

$$\therefore \text{ Total distance in yd. } \frac{4x}{3}.$$

Exercises 3

Write down:

1. The number of pence in b/-.
2. The number of shillings in £x.
3. The number of pence in a florins.
4. The number of shillings in d pence.
5. The number of pounds in y shillings.
6. The number of half-crowns in y sixpences.
7. The number of inches in x yd.
8. The number of feet in z in.
9. The number of pounds in c cwt.
10. The number of tons in c cwt.
11. The number of minutes in x hr.
12. The number of gallons in q pints.
13. The number of centimetres in a metres.
14. The number of kilograms in k gm.

Exercises 4

1. If one apple costs $2d.$, what is the cost of x apples?
2. If x articles cost $10s.$, what is the cost of 1 article?

3. If x lb. of sugar cost b pence, what is the cost of 1 lb.? What is the cost of 1 oz. of sugar?

4. A boy has a pence and gains b pence. How much has he now in pence? Express this sum in shillings.

5. In a school of a pupils there are b girls. How many boys are there?

6. In a school there are x boys. The number of girls is y more than the number of boys. How many pupils are there in the school?

7. A boy is x years old at present. How old will he be in 10 years time?

8. A boy is 15 years old at present. How old was he a years ago?

9. A clock is y min. slow. If the time on the clock is 3.15, what is the correct time?

10. The total weight of a truck and the coal in it is a tons. If the truck weighs k cwt., what is the weight of the coal, in cwt.?

CHAPTER 2

FORMULÆ AND SUBSTITUTION

Example 1:

A rectangle 3 in. long and 2 in. broad has an area of 3×2 sq. in.

A rectangle 5 ft. long and 3 ft. broad has an area of 5×3 sq. ft.

A rectangle 6 cm. long and 4 cm. broad has an area of 6×4 sq. cm.

and so on.

All these statements can be summed up in the following statement:

A rectangle l units long and b units broad has an area of $l \times b$ sq. units.

If we put A to represent the number of sq. units in the area

then $$A = l \times b$$
or $$A = lb$$

This is a formula which enables us to calculate the area of any rectangle if we know its length and its breadth.

Example 2:

A man, walking at 3 m.p.h., will travel in 2 hr. a distance of 3×2 miles.

A train, travelling at 1000 yd. per min., will travel in 5 min. a distance of 1000×5 yd.

An aeroplane, travelling at 450 m.p.h., will travel in $\frac{1}{2}$ hr. a distance of $450 \times \frac{1}{2}$ miles,

and so on.

All these statements can be summed up in the following formula:

$$S = vt$$

where S represents the number of units of distance travelled (say in miles), v represents the number of units of speed (say in m.p.h.) and t represents the number of units of time (say in hr.).

Example 3: The formula $C = \frac{22}{7}d$ enables us to find the length (C) of the circumference of any circle, if we know the length (d) of its diameter, e.g.:

Find the length of the circumference of a circle if its diameter is 14 in. long.

$$C = \frac{22}{7}d$$

Substitute the number 14 for d.

$$\therefore\ C = \frac{22}{7} \times 14$$
$$= 44$$

$\therefore$ The length of the circumference = 44 in.

Example 4: If $A = \frac{1}{2}h(a + b)$, find A when $h = 8$, $a = 4$, $b = 3$.

Substituting 8 for h, 4 for a, 3 for b in the formula, we have

$$\begin{aligned} A &= \tfrac{1}{2} \cdot 8(4 + 3) \\ &= 4(7) \\ &= 28 \end{aligned}$$

Exercises 5

1. How many pence are there in £k?
If S stands for this number, give the formula connecting S and k.

2. Find a formula for reducing tons to cwt. Choose T for the number of tons, and C for the number of cwt.

3. Find a formula for reducing: (*a*) yards to feet; (*b*) yards to chains. Choose your own letters in each case.

4. A room is l ft. long and b ft. broad. What is its perimeter in feet? If this number is P, find the formula connecting P, l, b.

5. In a school there are a classes, each containing x boys, and b classes, each containing y girls. Find a formula for N, the total number of pupils in the school.

If in one school $a = 5$, $b = 7$, $x = 32$, $y = 30$, find the total number of pupils in the school.

If $a = 6$, find the values of:

6. $a - 3$
7. $3a + 4$
8. $\frac{a}{2}$
9. $14 - 2a$
10. $\frac{2}{3}a - 1$
11. $a \quad \frac{1}{a}$
12. $(a + 2)(8 - a)$
13. $\frac{a}{6} + 15$
14. $a^2 + 2a$
15. $3a^2 - 5a$
16. $a^3 - 2a$
17. $a(a + 2)(a - 4)$
18. $(a - 3)(a + 5)(a - 6)$
19. $\frac{3}{a} + \frac{a^2}{12}$
20. $(2a)^3$
21. $2a^3$

If $a = 3$, $b = 2$, find the value of:

22. $a - b$
23. $2ab$
24. $\frac{a}{2b}$
25. $2a - 3b$
26. $(2a + b)(2a - b)$
27. $\frac{1}{a} + \frac{1}{b}$
28. $2 \quad \frac{1}{a}$
29. $\frac{3a + 4b}{2b - a}$
30. $3a^2b^2$
31. $(3ab)^2$
32. $a^2 + 2ab + b^2$

If $x = 4$, $y = \frac{1}{2}$, $k = 0$, find the value of:

33. xy

34. xy^2

35. $3x - 4y$

36. $x^2 + 2kx + y^2$

37. $\dfrac{x + k}{x - y}$

38. $\dfrac{2kxy}{x + 2y}$

39. Complete the following table:

x	1	2	3	4
$3x$				
$3x - 2$				

40. Complete the following table:

x	0	1	2	3
x^2	0	1	4	9
$5x$	0	5	10	15
$x^2 + 5x$	0	6	14	24

41. If $v = u + ft$, find v if $u = 20$, $f = 3$, $t = 6$.

42. If $S = \dfrac{n(n + 1)(2n + 1)}{6}$, find S when $n = 12$.

43. If $A = \pi r(r + l)$, find A if $\pi = \frac{22}{7}$, $r = 5$, $l = 16$.

44. If $A = P\left(1 + \dfrac{r}{100}\right)$, find A when $P = 250$, $r = 4$.

CHAPTER 3

EASY EQUATIONS AND PROBLEMS

Example 1: When you double a certain number and add 6, the result is 20. What is the number?

Let x stand for the number.
Double the number $= 2x$.
Adding 6 to $2x$ we get $2x + 6$.
But the result is 20.

$$\therefore \quad 2x + 6 = 20$$
$$\text{But} \quad 14 + 6 = 20$$
$$\therefore \quad 2x = 14$$
$$\text{But} \quad 2 \times 7 = 14$$
$$\therefore \quad x = 7$$
$\therefore$ The number is 7.

Note The statement $2x + 6 = 20$, i.e. that the number $2x + 6$ is equal to 20 is called an *equation*. The number x, which at first we do not know, is called the *unknown*, and, when we try to find the number which x stands for, we are trying to *solve* the equation.

$x = 7$ is said to be the *solution* of the equation.

Example 2: One boy has four times as many marbles as a second boy. If the first boy has 15 more than the second, how many marbles has each boy?

Let x stand for the number of marbles the second boy has.
The first boy has 4 times as many.
$\therefore$ The first boy has $4x$ marbles.

The difference in the numbers of marbles is $4x - x$.

$$\text{But the difference} = 15$$
$$\therefore \quad 4x - x = 15$$
$$3x = 15$$

$$\text{But } 3 \times 5 = 15$$
$$\therefore \quad x = 5$$

$\therefore$ The second boy has 5 marbles.

$\therefore$ The first boy has $4x$ marbles, i.e. 4×5 marbles, i.e. 20 marbles.

Exercises 6

1. x is a number. I double it and add 3. The result is 9. Find x.

2. x is a number. I multiply it by 4 and subtract 2. The result is 14. Find x.

3. I think of a number. I multiply it by 3 and add 4. The result is 19. Find the number.

4. I think of a number. I divide it by 3 and subtract 2. The result is 1. Find the number.

5. When 5 is subtracted from 4 times a number, the result is 27. Find the number.

6. Tom has five times as many marbles as Jim has. Together they have 42 marbles. How many marbles has each?

7. One boy has three times as many marbles as another. If the first boy has 16 more than the second, how many has each?

8. Two boys have 5*s.* 6*d.* between them. The first has twice as much as the second. How much has each?

9. On Monday a boy earns a certain sum, on Tuesday he earns 6*s.* 6*d.*, and on Wednesday he earns the same as he did on Monday. He earns altogether 19*s.* How much did he earn on Monday?

10. When a certain number is multiplied by 5, and divided by 8, the result is $2\frac{1}{2}$. Find the number.

11. A father is three times as old as his son. The difference in their ages is 28 years. How old is the son?

12. I buy a certain number of oranges at 4*d.* each, and the same number of oranges at 3*d.* each. If the total cost is 5*s.* 10*d.*, how many oranges did I buy altogether?

13. A collection of money consisted of fifteen shillings, twenty-one sixpences and a certain number of half-crowns. The total sum was £2 10*s*. 6*d*. How many half-crowns were there?

14. I bought a certain number of jotters at 8*d*. each. I paid for them with a 10*s*. note and got 3*s*. 4*d*. change. How many jotters did I buy?

15. A box weighs 5 lb. when empty. It contains a certain number of tins, each weighing $\frac{1}{4}$ lb. If the total weight of the box and its contents is 1 stone, how many tins are in the box?

Example 1: Solve $\frac{3x}{4} - 5 = 1$

$$\frac{3x}{4} - 5 = 1$$

$$\text{But } 6 - 5 = 1$$

$$\therefore \quad \frac{3x}{4} = 6$$

$$\text{But } \frac{24}{4} = 6$$

$$\therefore \quad 3x = 24$$

$$\text{But } 3 \times 8 = 24$$

$$\therefore \quad x = 8$$

Exercises 7

Solve the following equations, explaining how the solutions are found:

1. $x + 3 = 10$
2. $x - 5 = 4$
3. $3x + 7 = 25$
4. $19 - 2x = 9$
5. $\frac{2x}{3} = 8$
6. $4x = 4$
7. $\frac{1}{4}x - 3 = 1$
8. $5x = 0$
9. $8x - 64 = 0$
10. $16 - 3y = 1$
11. $7 + \frac{1}{2}x = 9$
12. $\frac{1}{2}x - \frac{1}{4}x = 10$
13. $21 = 4x - 7$
14. $3(x - 5) = 21$
15. $\frac{8 + 3x}{4} = 8$

CHAPTER 4

THE FOUR RULES

Addition and Subtraction

$$5 \text{ dozen} + 3 \text{ dozen} = 8 \text{ dozen}$$

Using the letter d to stand for a dozen or 12, this becomes

$$5d + 3d = 8d$$

Similarly, if the letter s stands for a score or 20

$$6s - 4s = 2s$$

But $3d + 5s$ could not be simplified further unless we substituted 12 for d and 20 for s.

Hence $8d + 6s - 3d + 2s = 8d - 3d + 6s + 2s$
$= 5d + 8s$

$8d + 6s - 3d + 2s$ is said to be an *algebraic expression*.

So are $2x$, $3x - y$ and $3ay + 4b - 5c^2$.

$2x$ is an expression with one *term*, $2x$.

$3x - y$ is an expression with two terms, $3x$ and $-y$.

$3ay + 4b - 5c^2$ is an expression with three terms. $3ay$, $+4b$ and $-5c^2$.

In the term $2x$, 2 is called the numerical *coefficient* of x.

In the expression $8d + 6s - 3d + 2s$, $8d$ and $-3d$ are said to be *like* terms, since they differ only in their numerical coefficients. So also are $+6s$ and $+2s$ like terms.

In the expression $3ab - 2c + 5ab + 2c^2$

$3ab$ and $+5ab$ are like terms.

$3ab$, $-2c$, $+2c^2$ are *unlike* terms.

Note that $2ab$ and $3ba$ would be like terms, for just as $3 \times 7 = 7 \times 3$ so $a \times b = b \times a$, i.e. $ab = ba$.

Similarly, $abc = bac = cba$

Hence $3abc + 5bac - 6cba = 2abc$.

Exercises 8

Simplify:

1. $a + a$
2. $3a + 2a$
3. $5b - b$
4. $7b - \frac{1}{2}b$
5. $x + \dfrac{x}{3}$
6. $x + x \times 2$
7. $\frac{1}{2}x + \frac{1}{3}x$
8. $\dfrac{x}{3} + \dfrac{x}{4}$
9. $5x + 2x + 3$
10. $6x - 3x + 4x$
11. $3ab + 2ab - ba$
12. $5xyz - 3zyx + xzy$
13. $5a + 4b - 3a - 2b$
14. $4c + 3d - 2c + 3 - d$
15. $3x + 8y - 2x - 3y$
16. $6 + 5y + 3x - 5y$
17. $2(3x + x) - 7x + y - 4$
18. $3\frac{1}{2}x + 4 - 1\frac{1}{2}x - 1\frac{1}{2}$
19. $4a^2 + 2a - 3a^2 + 5a$
20. $5x^2 + 10x - 2x - x^2$
21. $4x + x^2 - x + 3x^2$
22. $4ab + 3bc - ab - 2bc$
23. $5 + 8x + 6x^3 - 4x + x^2$
24. $6a + b + 4c - 4a + 4b - 2c$
25. $3x^3 + 5x^2 + 6x - 4x^2 - 2x^3 + 7$
26. $5a^2 + 2ab - a^2 + b^2 - ab + 3b^2$
27. $x + x \times x + x \times x$
28. $a + a + a \ldots$ to 17 terms
29. $a - a + a - a \ldots$ to 18 terms
30. $3x + 3x + 3x \ldots$ to 10 terms
31. $\dfrac{x}{3} + \dfrac{x}{3} + \dfrac{x}{3} \ldots$ to 6 terms
32. $\dfrac{a}{n} + \dfrac{a}{n} + \dfrac{a}{n} \ldots$ to n terms

Multiplication

$$
\begin{aligned}
3x \times 4y &= 3 \times x \times 4 \times y \\
&= 3 \times 4 \times x \times y \\
&= 12xy
\end{aligned}
$$

$$a^2 = a \times a, \text{ and } a^3 = a \times a \times a$$

$$
\begin{aligned}
\therefore \quad a^2 \times a^3 &= a \times a \times a \times a \times a \\
&= a^5, \text{ i.e. } a^{2+3}
\end{aligned}
$$

Similarly, $a^4 \times a^5 = a^{4+5} = a^9$

and $a^3 \times a^4 \times a^5 = a^{3+4+5} = a^{12}$

Hence the rule: **To multiply powers of the same number, add the indices.**

Example 1:

$$
\begin{aligned}
3a^3 \times 5a^2 &= 3 \times a^3 \times 5 \times a^2 \\
&= 3 \times 5 \times a^3 \times a^2 \\
&= 15a^{3+2} \\
&= 15a^5
\end{aligned}
$$

Example 2:

$$
\begin{aligned}
4a^2b \times 3ab^3 &= 4 \times a^2 \times b \times 3 \times a \times b^3 \\
&= 4 \times 3 \times a^2 \times a \times b \times b^3 \\
&= 12a^3b^4
\end{aligned}
$$

Example 3:

$$
\begin{aligned}
2(a^2)^3 &= 2 \times a^2 \times a^2 \times a^2 \\
&= 2a^6
\end{aligned}
$$

Example 4:

$$
\begin{aligned}
(2a^2)^3 &= 2a^2 \times 2a^2 \times 2a^2 \\
&= 2 \times 2 \times 2 \times a^2 \times a^2 \times a^2 \\
&= 8a^6
\end{aligned}
$$

Exercises 9

Show that:

1. $a^4 \times a^3 = a^7$
2. $2a^2 \times 3a^4 = 6a^6$
3. $(3a^3)^2 = 9a^6$
4. $3(a^3)^2 = 3a^6$
5. $3a \times 2a^2 \times 3a^3 = 18a^6$
6. Multiply by a: $5a$, $3a^2$, $2b$, $5b^3$
7. Multiply by $3a$: $2a$, $3ab$, $5a^2$, $4b^2$

8. Multiply by $2a^2$: $\frac{1}{4}a$, $3a^2$, $\frac{1}{6}ab$, $\frac{2}{5}a^2b^2$.

9. Multiply by $\frac{1}{2}y$: $2x$, $\frac{1}{4}x^2$, $6xy$, $4\frac{x}{y}$

Simplify:

10. $a \times a \times a$ . . . to 5 factors

11. $3a \times 3a \times 3a$ . . . to 5 factors

12. $\frac{x}{2} \times \frac{x}{2} \times \frac{x}{2}$ . . . to 4 factors

13. $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$ . . . to 6 factors

14. $a^2 \times a^4$

15. $\frac{a}{3} \times \frac{a}{4}$

16. $\frac{3}{a} \times \frac{5}{a^2}$

17. $5x \times 7y$

18. $a^2 \times a^3 \times a^4$

19. $3x \times 2x \times 4x^3$

20. $2 \times a^2 \times 2a$

21. $6b^2 \times \frac{1}{3}b$

22. $3a \times 4ab$

23. $\frac{3}{2}a \times 6a^2b$

24. $(3x)^2$

25. $3(x)^2$

26. $\left(\frac{5x}{2}\right)^2$

27. $5(x)^4$

28. $(2xy)^2$

29. $(\frac{2}{3}a^2b)^2$

30. $\frac{2}{3}(a^2b)^2$

31. $(x)(xy)(xyz)$

32. $(2x)(3xy)(4x^2y^2)$

33. $ab \times bc \times ca$

34. $abc \times bca \times cab$

35. $(4a^2bc)(5ab^2c)$

36. $3x^2y \times 2xy \times 3y$

37. $(a^2b)(ab^2)(a^2b^2c^2)$

38. $(ab)^2 \times ab^2$

39. $\left(\frac{3ab}{2}\right)^2 \times \frac{4}{3}a^2$

40. $\frac{1}{2}x^2y \times yz \times 4xyz^2$

Division

$$a^5 \div a^2 = \frac{\overset{1}{\cancel{a}} \times \overset{1}{\cancel{a}} \times a \times a \times a}{\underset{1}{\cancel{a}} \times \underset{1}{\cancel{a}}} = a \times a \times a = a^3, \text{ i.e. } a^{5-2}$$

Similarly, it can be shown that

$$a^7 \div a^2 = a^5, \text{ i.e. } a^{7-2}$$

Hence the rule: **To divide one power of a number by another power of the same number, subtract the indices.**

Example 1:

$$\frac{10a^6}{5a^2} = \frac{\overset{2}{\cancel{10}} \times a^6}{\underset{1}{\cancel{5}} \times a^2} = 2a^{6-2}, \text{ i.e. } 2a^4$$

Example 2:

$$\frac{12b^2c}{9bc} = \frac{\overset{4}{\cancel{12}} \times b^2 \times \overset{1}{\cancel{c}}}{\underset{3}{\cancel{9}} \times b \times \underset{1}{\cancel{c}}} = \frac{4 \times b^2}{3 \times b} = \frac{4}{3}b^{2-1} = \frac{4}{3}b$$

Example 3:

$$\frac{16a^4b^2c^3}{6a^2b^3c^3} = \frac{\overset{8}{\cancel{16}} \times a^4 \times b^2 \times \overset{1}{\cancel{c^3}}}{\underset{3}{\cancel{6}} \times a^2 \times b^3 \times \underset{1}{\cancel{c^3}}} = \frac{8 \times a^{4-2}}{3 \times b^{3-2}} = \frac{8a^2}{3b}$$

Exercises 10

Show that:

1. $a^8 \div a^4 = a^4$
2. $16a^5 \div 4a^2 = 4a^3$
3. $24a^2b^3c \div 8abc = 3ab^2$
4. $(\frac{1}{3}xy)^2 \div \frac{1}{9}x^2y^2 = 1$
5. Divide by a: $1, 5a, 6a^2, 3b, 2ab^2$
6. Divide by $3a^2$: $6a^2, 2a, 9a^2b, \frac{3}{a}, \frac{a}{2}$
7. Divide by $\frac{1}{2}a$: $1, a, x, \frac{2}{a}, \frac{a}{3}$

Simplify:

8. $a^7 \div a^2$
9. $a^4 \div a^4$
10. $a^6 \div a^3$
11. $6a^5 \div 2a^3$
12. $3a^6 \div 6a^4$
13. $\dfrac{(3a)^2}{2a^2}$
14. $\dfrac{3a^2}{(2a)^2}$
15. $\dfrac{16a^2}{4a^2}$
16. $5a^2 \div \dfrac{a}{4}$
17. $5xy \div xy$
18. $a^4b^4 \div a^2b$
19. $8a^3b^2c^3 \div 2ab^2c$
20. $\frac{1}{8}a^3b^2c \div \frac{1}{4}abc$
21. $2a^2 \times 6a^3 \div 4a^4$
22. $ab \times bc \times ca \div abc$
23. $\dfrac{(3x^2)(18y^4)}{6xy^2}$
24. $\dfrac{12x^3d^3}{2cd^2 \times 4d}$
25. $\dfrac{ab \times (bc)^2}{(ab)^2 \times c}$
26. $\dfrac{6a^2b^3 \times a}{9ab^2 \times 2b}$
27. $\dfrac{(20ab^2)(3a^2b)}{15a^4b^4}$
28. $\dfrac{2xyz \times 3x^2yz^2 \times xy^2z}{9x^3y^3}$

REVISION PAPERS 1–5

Paper 1

1. Write down, without simplifying, using only letters and signs:

(i) twice the sum of a and b;
(ii) the sum of a and twice b.

Find the value of each answer when $a = 4$, $b = 5$.

2. When $x = 3$ find the value of $3x^2 - 4x + 5$

3. Solve the equation $6x - 9 = 39$.

4. Simplify:

$$6a + 3b + 5 - 2b - a - 2$$

5. Write in shorter form:

(i) $a + a + a + a$	(ii) $a \times a \times a \times a$
(iii) $2a \times 2a \times 2a$	(iv) $2 \times a \times a \times a$
(v) $aaaa^3$	(vi) $\frac{xxx}{yy}$

6. A boy bought a jotters at $8d.$ each and 8 jotters at $\frac{a}{2}$ pence each. Find the total cost in shillings.

Paper 2

1. x and y are two numbers, x being less than y. Write down expressions for: the square of their difference, twice the square of their sum, their difference divided by twice their product.

2. Simplify:

$$6x + 5y + 2x - 3y - 5x - 2y$$

3. Solve $$\frac{1+x}{3} - 4 = 0$$

4. When $a = 3$, $b = 2$ find the value of
$$(a^3 - b^3) - (a - b)^3$$

5. Simplify:

(i) $3x^2 \times 4x^3$ (ii) $\dfrac{16a^3}{4a^2}$

(iii) $9a^3b \div 3a^2b$ (iv) $\sqrt{16a^4}$

6. Write down expressions for the following:

(i) cost in shillings of $4a$ articles at $3b$ pence each;

(ii) number of bottles required for 3 gal., if each bottle holds k pints;

(iii) value in pounds of x half-crowns, y florins, z shillings;

(iv) the number which exceeds 5 by as much as 5 exceeds k.

Paper 3

1. When 4 is added to five-sixths of x, the result is 29. Find x.

2. A car travels $4x$ miles in 5 min. How many miles does it travel in 1 hr.? How many minutes will it take to travel $(8x^2 + 4x)$ miles?

3. If $x = 2a$, $y = 5a$, find the value of:

(i) $2x + y$ (ii) xy (iii) $2x^2y$ (iv) $\dfrac{2x + 8y}{10y - 3x}$

4. Simplify:
$$5a^2 + 3a + 4 - 2a + a^2 - 1 - a + 2$$

5. Simplify:

(i) $2x^2y \times 5xy^3$ (ii) $24x^3y^2 \div 6x^2y$

(iii) $\frac{1}{2}x^2y^2 \div \frac{1}{4}xy^2$

6. The marked price of an article is $12q$ shillings, and a discount of a pence per shilling is given during a sale. What is the actual selling price? If the marked price of the article was 36*s.*, what was the discount?

Paper 4

1. If $a = 2$, $b = 1$, $c = 0$, find the value of:

(i) $3a^2 - 2b^2 + 2c^2$ (ii) $ab + bc + ca$

(iii) $\dfrac{9(a - b)^3}{2(a + b)^2}$

2. If $12 - 3t = 6$, find the value of t. Hence find the value of:

(i) $\dfrac{t}{4} + 1$ (ii) $t^3 - 2t^2$

3. If $x = 6a$, $y = 2a$, find the value of

$$5(x - 2y) - 2(5y - x)$$

If the answer exceeds a by 7, find the value of a.

4. Simplify:

$$4x^2 + 5xy + 7y^2 - 3xy + x^2 - y^2 + xy - 5x^2$$

5. Simplify:

(i) $\frac{3}{4}a^2b \times \frac{2}{3}ab^3$ (ii) $\left(\dfrac{x^2y}{3}\right)^2 \times 3xy^2$

(iii) $\sqrt{81a^2b^4}$ (iv) $\dfrac{24a^2b^2c}{4ab^2}$

6. y years ago a man was y years old. His son is y years younger than his father. How old is the son at present? How old was he 3 years ago? If at that time the son was 22 years old, find y.

Paper 5

1. If a tons b cwt. coal cost x pounds y shillings, find an expression for the cost of 1 cwt. in shillings. Find the cost of 1 cwt. if $a = b = 4$, $x = 35$, $y = 14$.

2. Solve $\frac{5}{6}x - \frac{1}{4}x = 7$.

3. A man works an 8-hr. day for 5 days a week, and during the week he works 8 hr. overtime. His regular wage is a shillings per hour, and he is paid time and a half for overtime. How many pounds does he earn that week?

4. A lorry can carry 6 tons of goods. It is loaded with 18 sacks, each weighing x cwt., and $8x$ boxes, each weighing 1 qr. How many more tons could it take? Find the cost of carriage for its present load at 6*d*. per ton per mile for a journey of 100 miles.

5. If $a^b = c$, find c^b when $a = b = 2$.

6. If $\frac{1}{x} + \frac{1}{a} = \frac{1}{b}$; find the value of x, if $a = 3$, $b = \frac{3}{4}$.

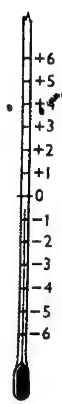
CHAPTER 5

DIRECTED NUMBERS

So far the only numbers we have been concerned with have been numbers such as 0, 1, $2\frac{1}{2}$, $3\frac{8}{11}$, 7, etc., and these have been sufficient for all our purposes.

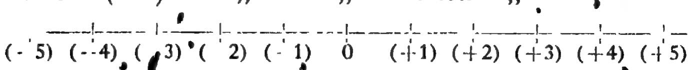
Answers to questions such as $6 - 4 = ?$, $5 - 5 = ?$ are easily found, but no answer meantime can be given to $4 - 6 = ?$.

We proceed to show how an answer can be found. So far the sign + denotes addition, and the sign − denotes subtraction, but there is another use for the signs + and −.



If we examine a Centigrade thermometer, we find that the Freezing Point of water is marked 0°. Temperatures *above* zero are marked by the numbers +1, +2, +3, etc., while those *below* zero are marked by the numbers −1, −2, −3, etc. The sign + directs the number upwards from zero, while the sign − directs the number downwards from zero. To avoid any confusion with the other use of + and −, brackets will be used meantime when the signs + and − indicate direction.

Thus $(+5)°$ will mean 5 degrees *above* zero.
while $(-5)°$,, ,, *below* ,,



In the same way, it has been agreed that, if we mark equal steps along a line, as shown above, and mark any point on

the line 0, steps to the *right* of 0 are regarded as + steps, and those to the *left* of 0 are regarded as — steps.

There is no real reason why steps upward and to the right should be regarded as + steps, while those to the left and downward should be regarded as — steps. The opposite arrangement would have done quite as well, but the one already stated has been agreed on universally.

Addition of Directed Numbers

(−5) (−4) (−3) (−2) (−1) 0 (+1) (+2) (+3) (+4) (+5)

Example 1: Add (+3) and (+2).

We start at the point 0, move 3 steps to the right (+3), and then move a further 2 steps to the right (+2). The new position is 5 steps to the right of 0, i.e. (+5).

$$\therefore (+3) + (+2) = (+5)$$

Example 2: Add (+3) and (−5).

We start at 0 and move 3 steps to the right (+3), and then move 5 steps to the left (−5). The new position is 2 steps to the left of 0, i.e. (−2).

$$\therefore (+3) + (-5) = (-2)$$

Example 3: Add (−3) and (+5).

We start at 0 and move 3 steps to the left (−3), and then move 5 steps to the right (+5). The new position is 2 steps to the right of 0, i.e. (+2).

$$\therefore (-3) + (+5) = (+2)$$

Example 4: Add (−3) and (−2).

We start at 0 and move 3 steps to the left (−3), and then move a further 2 steps to the left (−2). The new position is 5 steps to the left of 0, i.e. (−5).

$$\therefore (-3) + (-2) = (-5)$$

From the above examples it is clear that the rules for adding two directed numbers are:

1. When the signs are like (i.e. both + or both −), add and prefix the same sign.

2. When the signs are unlike (i.e. one +, the other −), find the difference and prefix the sign of the larger.

$$(+6x) + (-2x) = (+4x)$$

Here the signs are unlike. We find the difference between $6x$ and $2x$, namely $4x$, and prefix the sign of the larger, namely +.

Similarly, $(-3a) + (-7a) = (-10a)$
$(-4y) + (+2y) = (-2y)$

Three or more directed numbers are added as follows:

$$\begin{aligned} & (+a) + (-5a) + (+6a) + (-3a) \\ = & (+a) + (+6a) + (-5a) + (-3a) \\ = & (+7a) + (-8a) \\ = & (-a) \end{aligned}$$

Exercises 11

Write down answers to the following:

1. $(+6) + (+2)$ **4.** $(+3) + (-3)$ **7.** $(-4) + (-5)$
2. $(+5) + (+1)$ **5.** $(-5) + (+8)$ **8.** $0 + (-8)$
3. $(+3) + (-6)$ **6.** $(-6) + (+2)$ **9.** $(-7) + (+7)$
10. $(+2) + (+3) + (+4)$
11. $(-2) + (+3) + (-5)$
12. $(+2) + (+1) + 0$
13. $(-6) + (+2) + (-3)$
14. $(+2) + (+4) + (-3) + (+1)$
15. $(+2a) + (+4a)$
16. $(+6b) + (-3b)$
17. $(-3x) + (-2x)$
18. $(-2x^2) + (+2x^2)$
19. $(+5ab) + (-6ab)$

20. $(-8abc) + (+2abc)$

21. $(+a) + (+4a) + (-2a)$

22. $(+3x) + (-6x) + (+2x)$

23. $(+5a^2) + (-3a^2) + (-2a^2)$

24. $(-2b) + (-5b) + (-3b)$

25. $(-6x) + (+x) + (-2x)$

26. $(+4c) + (-3c) + (-c)$

27. $(+5x^2) + (-x^2) + (+3x^2) + (-2x^2)$

28. $(+ab) + (-ab) + (-ab) + (-ab)$

29. $(-5ab) + (+4ab) + (-2ab) + (-ab)$

30. $(-3xy) + (+2xy) + (+6xy) + (-4xy)$

Subtraction of Directed Numbers

When subtracting 4 from 7, we ask ourselves what number must be added to 4 to give 7.

Thus $7 - 4 = 3$, since 3 must be added to 4 to give 7.

When subtracting directed numbers we proceed in the same way.

Example 1: $(+5) - (+3)$.

Since $(+2)$ must be added to $(+3)$ to give $(+5)$,

$$\therefore\ (+5) - (+3) = (+2)$$

Example 2: $(+3) - (+5)$.

Since (-2) must be added to $(+5)$ to give $(+3)$,

$$\therefore\ (+3) - (+5) = (-2)$$

Example 3: $(-9) - (+2)$.

Since (-11) must be added to $(+2)$ to give (-9),

$$\therefore\ (-9) - (+2) = (-11)$$

Example 4: $(-9)-(-2)$.
Since (-7) must be added to (-2) to give (-9),

$$\therefore\ (-9)-(-2)=(-7)$$

We have seen in Example 1 that

$$\begin{aligned} & (+5)-(+3)=(+2) \\ \text{But}\ & (+5)+(-3)=(+2) \\ \therefore\ & (+5)-(+3)=(+5)+(-3) \end{aligned}$$

Hence, if in the subtraction sum we change the sign of the number to be subtracted and then add, we get the correct answer. Thus,

Example 2
$(+3)-(+5)$ becomes $(+3)+(-5)=(-2)$

Example 3
$(-9)-(+2)$ becomes $(-9)+(-2)=(-11)$

Example 4
$(-9)-(-2)$ becomes $(-9)+(+2)=(-7)$

Hence the rule for subtracting two directed numbers is: **Change the sign of the number to be subtracted and add.**

Thus,

$$\begin{aligned} (-6x)-(-2x) &= (-6x)+(+2x) \\ &= (-4x) \\ (-3a)-(-7a) &= (-3a)+(+7a) \\ &= (+4a) \\ (-4y)-(+2y) &= (-4y)+(-2y) \\ &= (-6y) \end{aligned}$$

Similarly,

$$\begin{aligned} (+3a)-(-4a)-(+2a) &= (+3a)+(+4a)+(-2a) \\ &= (+7a)+(-2a) \\ &= (+5a) \end{aligned}$$

Exercises 12

What must be added to:

1. $(+3)$ to give $(+7)$?
2. (-3) to give $(+5)$?
3. $(+4)$ to give (-6)?
4. $(-3a)$ to give $(-7a)$?
5. $(-4b)$ to give 0?

Write down answers to the following:

6. $(+9)-(-3)$
7. $(-3)-(-6)$
8. $(+4)-(+7)$
9. $(-9)-(+4)$
10. $0-(+6)$
11. $(-5)-0$
12. $(+5x)-(+2x)$
13. $(-6a)-(+3a)$
14. $(+4a)-(-2a)$
15. $(-5b)-(-4b)$
16. $(-3x)-(-3x)$
17. $(+7a^2)-(-3a^2)$
18. $(-7ab)-(+5ab)$
19. $(+3x^2)-(+6x^2)$
20. $(+6y)-(-4y)$
21. $(-5abc)-(-abc)$
22. $(+2x)-(+x)-(+3x)$
23. $(+3y)-(-2y)-(-4y)$
24. $(-5a)-(-3a)-(+2a)$
25. $(-6x^2)+(+2x^2)-(+x^2)$
26. $(-6ab)+(-5ab)-(-ab)$
27. $(-3a^2)-(-2a^2)+(-a^2)$
28. $(-a)+(+a)-(+a)-(-a)$
29. $(-3b)-(-5b)+(+2b)+(-b)$
30. $(+5x)-(+3x)-(-2x)-(-x)$

Multiplication of Directed Numbers

West East

(-200) (-150) (-100) (-50) 0 $(+50)$ $(+100)$ $(+150)$ $(+200)$

Let the above line represent a railway running East and West through a place situated at 0. The distances east of 0 are regarded as + (positive) and the distances west of 0 are regarded as − (negative).

I. Consider a train travelling *east* (i.e. in the positive direction) at 50 m.p.h. Its velocity is $(+50)$ m.p.h. Let it reach 0 at noon. Let times before noon be regarded as negative and times after noon be regarded as positive. Thus (-3) hours would be 3 hours before noon, and $(+2)$ hours would be 2 hours after noon.

(1) 4 hours after noon the distance of the train from 0 would be 200 miles *east* of 0, i.e. (+200) miles. But the distance in miles is got by multiplying the velocity of the train (+50) by the number of hours (+4),

$$\therefore\ (+50) \times (+4) = (+200)$$

(2) 4 hours before noon the train would be 200 miles *west* of 0, i.e. at the point (−200) miles.

But the distance = (+50) × (−4) miles,

$$\therefore\ (+50) \times (-4) = (-200)$$

II. Consider a train travelling *west* (i.e. in the negative direction) at 50 m.p.h., i.e. its velocity is (−50) m.p.h. Let it reach 0 at noon. Let times before noon be regarded as negative and times after noon be regarded as positive.

(3) 4 hours after noon the train is 200 miles *west* of 0, i.e. at the point (−200) miles.

But the distance = (−50) × (+4) miles,

$$\therefore\ (-50) \times (+4) = (-200)$$

(4) 4 hours before noon the train is 200 miles *east* of 0, i.e. at the point (+200) miles.

But the distance = (−50) × (−4) miles,

$$\therefore\ (-50) \times (-4) = (+200)$$

Similarly,
$$\begin{aligned}(+a) \times (+b) &= (+ab)\\ (+a) \times (-b) &= (-ab)\\ (-a) \times (+b) &= (-ab)\\ (-a) \times (-b) &= (+ab)\end{aligned}$$

Hence the rule: **In multiplying two directed numbers, like signs give +, unlike signs give −.**

Continued Products

$$(+a) \times (+b) \times (+c) = (+ab) \times (+c) = (+abc)$$
$$(-a) \times (-b) \times (-c) = (+ab) \times (-c) = (-abc)$$

$$(-a) \times (+b) \times (-c) = (-ab) \times (-c) = (+abc)$$
$$(+a) \times (-b) \times (+c) = (-ab) \times (+c) = (-abc)$$
$$(-a) \times (-b) \times (-c) \times (+d) = (+ab) \times (-cd) = (-abcd)$$
$$(-a) \times (+b) \times (-c) \times (+d) = (-ab) \times (-cd) = (+abcd)$$

Hence:

(1) If the signs are all positive, the product is positive.

(2) If there is an *odd* number of negative signs, the product is negative.

(3) If there is an *even* number of negative signs, the product is positive.

Exercises 13

1. $(+3) \times (+5)$
2. $(+7) \times (-3)$
3. $(-4) \times (+4)$
4. $(-3) \times (-6)$
5. $(-1) \times (-1)$
6. $0 \times (+5)$
7. $(-3) \times 0$
8. $(+2) \times (-7)$
9. $(-3) \times (-4)$
10. $(+6) \times (+5)$
11. $(-8) \times (+2)$
12. $(-2) \times (-6)$
13. $(+2) \times (-3) \times (-1)$
14. $(-1) \times (-1) \times (-1)$
15. $(-2) \times (+5) \times (+3)$
16. $(+3) \times (-2) \times (-5)$
17. $(+1) \times (-1) \times (-1) \times (-1)$
18. $(-1) \times (-2) \times (-3) \times (-4)$
19. $(-2) \times (+3) \times (+1) \times (+4)$
20. $(+2) \times (\ \ 1) \times (-5) \times (-2) \times (-3)$
21. $(+3a) \times (+2)$
22. $(-4a) \times (+3)$
23. $(+5a) \times (-2)$
24. $(-3x) \times (-4)$
25. $(-5t) \times (+4)$
26. $(-a) \times (+b)$
27. $(-m) \times (-n)$
28. $(+x) \times (-y)$
29. $(+3x) \times (+y)$
30. $(-4a) \times (+2b)$
31. $(-3a) \times (-2b)$
32. $(+2m) \times (-3n)$
33. $(-3x) \times (+4y)$
34. $(+x)(+y)(-z)$

35. $(-a)(-b)(-c)$
36. $(-r)(+s)(-t)$
37. $(+2a)(-b)(-2c)$
38. $(-2x)(-3y)(+z)$
39. $(+4x)(-5y)(+3z)$
40. $(+a)(-b)(-c)(+d)$
41. $(-5b)(-1)(+2a)$
42. $(-x)(+x)(-x)$
43. $(-a)(-a)(+a)(-a)$
44. $(-2x)(-2x)(+2x)$

Powers of Directed Numbers

$$(+a)^3 = (+a)(+a)(+a) = +a^3$$
$$(+a)^6 = (+a)(+a)(+a)(+a)(+a)(+a) = +a^6$$
$$(-a)^2 = (-a)(-a) = +a^2$$
$$(-a)^3 = (-a)(-a)(-a) = -a^3$$
$$(-a)^4 = (-a)(-a)(-a)(-a) = +a^4$$

Hence:

(1) **all powers of positive numbers are positive;**
(2) **even ,, negative ,, positive;**
(3) **odd ,, negative ,, negative.**

The following examples should be noted carefully:

$$2(-a)^3 = 2 \times (-a^3) = -2a^3$$
$$(-2a)^3 = -8a^3$$
$$3(-a)^4 = 3 \times (+a^4) = 3a^4$$
$$(-3a)^4 = 81a^4$$

Exercises 14

1. $(+2)^2$
2. $(-3)^2$
3. $(+3)^3$
4. $(-1)^4$
5. $(-2)^3$
6. $(-1)^5$
7. $(+2)^6$
8. $(-1)^3$
9. $(+3)^4$
10. $(-2)^2$
11. $(+a) \times (+a) \times (+a)$
12. $(-b) \times (-b) \times (-b) \times (-b)$
13. $(-x)(-x)(-x)$
14. $(-a)(-a)(-a)(-a)(-a)$
15. $(+a)^3$
16. $(-a)^2$
17. $(+x)^4$
18. $(-a)^4$

19. $(-m)^5$
20. $(+y)^2$
21. $(+x)^5$
22. $\left(-\dfrac{x}{2}\right)^2$
23. $\left(+\dfrac{a}{3}\right)^3$
24. $\left(-\dfrac{a}{2}\right)^4$
25. $2(-a)^3$
26. $(-2a)^3$
27. $(-3x)^2$
28. $3(-x)^2$
29. $(+2x)^3$
30. $2(+a)^4$
31. $x(+x)^2$
32. $a(-a)^3$
33. $a(+2a)^2$
34. $\left(-\dfrac{a^2}{3}\right)^2$
35. $(-ab) \times (+a)$
36. $(-xy) \times (-xy)$
37. $(+3a) \times 0 \times (-2a)$
38. $(-x)^2 \times (-y)^2$
39. $x(-y)^2$
40. $x(-y)^3$
41. $(+ab)^2$
42. $2(-ab)^3$
43. $(-a)^2 \times (-a^2)$
44. $(-x^2y)(xy^2)$
45. $(-x)^2(-y)^2$
46. $(-a)^3(-a)^2$
47. $(+3x)(-xy)$
48. $(-ab)(+a)(-b)$
49. $(-5b)(-1)(+2b)$
50. $(-a^2)^3$
51. $\left(-\quad 2b\right)$

Division of Directed Numbers

When dividing 15 by 3 we ask ourselves what number must be multiplied by 3 to give 15, and since 5 is the number

$$15 \div 3 = 5.$$

We proceed in the same way in the division of directed numbers.

Example 1: $(+15) \div (+3)$.

Since $(+3)$ has to be multiplied by $(+5)$ to give $(+15)$,

$$\therefore\ (+15) \div (+3) = (+5)$$

Example 2: $(+15) \div (-3)$.
Since (-3) has to be multiplied by (-5) to give $(+15)$,

$$\therefore\ (+15) \div (-3) = (-5)$$

Example 3: $(-15) \div (+3)$.
Since $(+3)$ has to be multiplied by (-5) to give (-15),

$$\therefore\ (-15) \div (+3) = (-5)$$

Example 4: $(-15) \div (-3)$.
Since (-3) has to be multiplied by $(+5)$ to give (-15),

$$\therefore\ (-15) \div (-3) = (+5)$$

Hence, in the division of one directed number by another, the quotient is positive if the signs are like, and negative if the signs are unlike.

Therefore in both multiplication and division:

Like signs give plus, unlike signs give minus.

Exercises 15

Simplify:

1. $(+9) \div (+3)$
2. $(+16) \div (-4)$
3. $(-16) \div (+2)$
4. $(-12) \div (-3)$
5. $0 \div (+5)$
6. $(-10) \div (-1)$
7. $(+3x) \div (+3)$
8. $(-14a) \div (-2)$
9. $(+10b) \div (-5)$
10. $0 \div (-3x)$
11. $(+12a) \div (+6a)$
12. $(-8a) \div (+4a)$
13. $(+15x) \div (-5x)$
14. $(-18y) \div (+3y)$
15. $(+6ab) \div (-2a)$
16. $(+pq) \div (-pq)$
17. $(+a^3) \div (+a)$
18. $(-8a^3) \div (-2a)$
19. $(-16a^2b) \div (+4ab)$
20. $0 \div (-2x^2)$
21. $(-20xy) \div (+4y)$
22. $(-14ab) \div (-7)$
23. $(+8xy) \div (-4xy)$
24. $(+6a^2) \div (+2a)$
25. $(-21x^2) \div (+3x)$
26. $(+x^4) \div (-x^3)$
27. $(-x^5) \div (-x^2)$
28. $(-2y^6) \div (+y^4)$

29. $\dfrac{(+x^3y^2)}{(-xy^2)}$

30. $\dfrac{(-a^2b^3)}{(-ab)}$

31. $\dfrac{2(-m)^3}{8m^2}$

32. $\dfrac{(-2m)^3}{-4m}$

33. $\dfrac{(+2ab)^3}{4a^2b}$

34. $\dfrac{-2(-3x^2)^2}{12x^3}$

In the addition of directed numbers it has been shown that

$$(+3) + (+2) = (+5)$$
$$\text{But } +3 + 2 = +5$$
$$\therefore\ (+3) + (+2) = +3 + 2$$

Similarly,

$$(+3) + (-2) = +1$$
$$\text{But } +3 - 2 = +1$$
$$\therefore\ (+3) + (-2) = +3 - 2$$

$$\text{Also } (-3) + (+2) = -1$$
$$\text{But } -3 + 2 = -1$$
$$\therefore\ (-3) + (+2) = -3 + 2$$

$$(-3) + (-2) = -5$$
$$\text{But } -3 - 2 = -5$$
$$\therefore\ (-3) + (-2) = -3 - 2$$

Hence in the addition of directed numbers we can omit the + sign connecting the directed numbers, write the numbers without brackets and deal with the numbers in accordance with their own sign.

Note that $(+3y)$ may be written as $+3y$ or $3y$.
$(-3y)$ is written as $-3y$.

In the subtraction of directed numbers it has been seen that

$(+5) - (+3) = (+5) + (-3) = +5 - 3$ (as shown above)

Similarly,

$$(+3) - (+5) = (+3) + (-5) = +3 - 5 \quad \text{(as shown above)}$$

$$(-9) - (+2) = (-9) + (-2) = -9 - 2 \quad \text{(as shown above)}$$

$$(-9) - (-2) = (-9) + (+2) = -9 + 2 \quad \text{(as shown above)}$$

Hence in the subtraction of directed numbers we can omit the — sign connecting the directed numbers, and write the numbers without brackets, provided we change the sign of the number to be subtracted. Thus,

$$(-6x) - (-3x) = -6x + 3x = -3x$$

Example: Simplify $2a - 3b + 6b - 7a - 3c$

$$2a - 3b + 6b - 7a - 3c = +2a - 7a - 3b + 6b - 3c$$
$$= -5a + 3b - 3c$$

(since $+2a - 7a = -5a$ and $-3b + 6b = +3b$)

In the multiplication and division of directed numbers we can also drop the brackets and deal with the numbers in accordance with their own signs, as the following examples show:

Example 1: $(-3xy) \times (-5x^2)$ may be written

$$-3xy \times -5x^2 = +15x^3y$$

Example 2: $(+12a^3b^2) \div (-3ab)$ may be written

$$+12a^3b^2 \div -3ab = -4a^2b$$

Hence, when dealing in future with directed numbers, we can omit the brackets and no confusion will arise.

Exercises 16

Simplify:

1. $+3a + 5a$
2. $+5b - 2b$
3. $+6c - 8c$
4. $-5a + 6a$
5. $-7a + 7a$
6. $-4x - 5x$
7. $-8b + 3b$
8. $5a - 2a + 4b$
9. $-6a + 3b - 2b$
10. $4a + 6b - 3a - 2b$
11. $-2a + 5b - a + 2b$
12. $3a - 3b + 2a + 4b$
13. $2x^2 - 3x - 3x^2 + 5x$
14. $-x^2 + 2x - 3x + 5x^2$
15. $-4a + 2b - c + 3a - b$
16. $3x^2 + 5x - 6 - 7x - 5x^2$
17. $9a^2 - 3 - 4a^2 + 6a - 5$
18. $x^2 - 2x + 3x - 6 + 2x^2 - 4$
19. $a + b - c - 2a + 3b + 4c$
20. $-2a^2 + 3b^2 - 5ab - 6b^2 + 2ab$
21. $+2a \times -3$
22. $-3x \times -2$
23. $-5a \times +3$
24. $+2x \times +3x$
25. $-3a^2 \times -5a$
26. $+6a \div +3$
27. $-8a \div -4$
28. $-12x \div +2$
29. $+9a \div +3a$
30. $-8a^2 \div -4a$

Chapter 6

BRACKETS I

We have already seen that numbers inside brackets are to be regarded as single numbers, e.g. $10 + (5 + 2)$ means that 2 more than 5 is to be added to 10. Hence, after adding 5 to 10 we must also add 2.

$$\therefore \quad 10 + (5 + 2) = 10 + 5 + 2 \quad . \quad . \quad (1)$$

Similarly, $10 + (5 - 2)$ means that 2 less than 5 is to be added to 10. Hence, after adding 5 to 10 we must subtract 2.

$$\therefore \quad 10 + (5 - 2) = 10 + 5 - 2 \quad . \quad . \quad (2)$$

Again, $10 - (5 + 2)$ means that 2 more than 5 is to be subtracted from 10. Hence, after subtracting 5 from 10 we must also subtract 2.

$$\therefore \quad 10 - (5 + 2) = 10 - 5 - 2 \quad . \quad . \quad (3)$$

and $10 - (5 - 2)$ means that 2 less than 5 is to be subtracted from 10. Hence after subtracting 5 from 10, we must add 2.

$$\therefore \quad 10 - (5 - 2) = 10 - 5 + 2 \quad . \quad . \quad (4)$$

The results in (1), (2), (3) and (4) can be stated generally as follows:

$$\begin{aligned} a + (b + c) &= a + b + c \\ a + (b - c) &= a + b - c \\ a - (b + c) &= a - b - c \\ a - (b - c) &= a - b + c \end{aligned}$$

Hence the following rules:

1. When a + sign precedes a bracket the brackets may be removed if the sign of each of its terms is *not* changed.

2. When a − sign precedes a bracket the bracket may be removed, if the sign of each of its terms is changed.

Example 1: Simplify $(3a + 7b) - (2a - 4b)$.

Consider the first part of the expression $+(3a + 7b)$. Inside the bracket there are two terms, $+3a$ and $+7b$. Since the sign preceding the bracket is $+$, the bracket may be removed, if we do not change the signs of these terms.

$$\therefore \ +(3a + 7b) = +3a + 7b$$

Consider the second part of the expression $-(2a - 4b)$. Inside the bracket there are two terms, $+2a$ and $-4b$. Since the sign preceding the bracket is $-$, the bracket may be removed, if we change the sign of each of these terms.

$$\therefore \ -(2a - 4b) = -2a + 4b$$

Hence,
$$\begin{aligned} (3a + 7b) - (2a - 4b) &= +3a + 7b - 2a + 4b \\ &= +3a - 2a + 7b + 4b \\ &= a + 11b \end{aligned}$$

Example 2: Simplify:

$$-(3x^2 + 7x - 4) - (-2x^2 - 5x + 3) - (7x + 2)$$

$$\begin{aligned} \text{Expression} &= -3x^2 - 7x + 4 + 2x^2 + 5x - 3 - 7x - 2 \\ &= -3x^2 + 2x^2 - 7x + 5x - 7x + 4 - 3 - 2 \\ &= -x^2 - 9x - 1 \end{aligned}$$

Exercises 17

Simplify:

1. $a + (a - b)$
2. $a - (2a + b)$.
3. $-(x + y) - 2x$
4. $(a - b) - (a + b)$
5. $-(a - b) + (2a - 3b)$
6. $(x - 1) - (1 + x)$

7. $1 - (x - 1)$
8. $3a + (3b - 4a)$
9. $4b - (3 - b)$
10. $-(2x + 3y) + 6x$
11. $(2a + 4b) - (3a - b)$
12. $(p + 2q) - (-p - q)$
13. $-(3x - 2y) + (3y - 2x)$
14. $(1 - x) + (x - 1) - 1$
15. $(2x + 3y) - (3x - 2y) + 5x$
16. $(2 + x) + (x - 2) - (2x - 1)$
17. $(2x^2 - 3x) - (5x + 2) + (x^2 - 3x)$
18. $(x - y - z) - (y - z + x)$
19. $-(x - y) - (y - z) - (z - x)$
20. $(a + b - c) + (b + c - a) - (c + a - b)$
21. $(6s - 4) + (8s - 3) - (4s - 5)$
22. $-(3a - 4b) + (5a - 3b) - (6a - 1)$
23. $(3x^2 + xy) - (2xy - 3y^2) - (x^2 - y^2)$
24. $6a^2 - (5b^2 + 3a^2) - (3a^2 - 5b^2)$
25. $2x^2 - (3x^2 - xy - y^2) - (3xy - 4x^2)$
26. $-(2a^2 - b^2 + 3c^2) + (b^2 - 2c^2 - 3a^2) - (c^2 - 3b^2 - a^2)$

Complete the following:

27. $a + b + c = a + (\qquad)$
28. $a + b - c = a + (\qquad)$
29. $a - b + c = a - (\qquad)$
30. $a - b - c = a - (\qquad)$
31. $-3a + 2b - c = -(\qquad) - c$
32. $x - y - z = -(\qquad)$

Using brackets, write the following statements, and simplify:

33. Subtract $3a - 2b$ from $a + b$.

34. From the sum of $2x + 3y$ and $2y - 6x$ subtract $x - y$.

35. Find the sum of $5a - 7b$ and $2b - 3a$, and the sum of $6a - 6b$ and $b - a$. Subtract the first result from the second.

We know that

$$3(5 + 4) = 3(9) = 27.$$

But $$3 \times 5 + 3 \times 4 = 15 + 12 = 27$$
$$\therefore\ 3(5 + 4) = 3 \times 5 + 3 \times 4$$

Similarly, $a(b + c)$ means that each of the terms inside the bracket is to be multiplied by a, i.e. $+b$ and $+c$ are each to be multiplied by $+a$.

$$+b \times +a = +ab \text{ and } +c \times +a = +ac$$
$$\therefore\ a(b + c) = ab + ac$$

In the same way

$$\begin{aligned} a(b - c) &= ab - ac \\ -a(b + c) &= -ab - ac \\ -a(b - c) &= -ab + ac \\ x(x + y - z) &= xx + xy - xz \\ &= x^2 + xy - xz \end{aligned}$$

Example. Simplify

$$-3x(x - 2y) - y(x - 3y) + (x^2 - 3y^2)$$

$$\begin{aligned} \text{Expression} &= -3x^2 + 6xy - yx + 3y^2 + x^2 - 3y^2 \\ &= -3x^2 + x^2 + 6xy - yx + 3y^2 - 3y^2 \\ &= -2x^2 + 5xy \end{aligned}$$

Exercises 18

Simplify:

1. $10 - 2(4 - 3)$
2. $2(a + b) - 4a$
3. $3(2a - b) + 4b$
4. $2(x + y) + 3(x - y)$
5. $3(x - 2y) - 4(2x - y)$
6. $\frac{1}{2}(4x + 6)$
7. $\frac{1}{3}(12y - 9)$
8. $\frac{1}{2}(2x - 4) - \frac{1}{4}(8x + 12)$
9. $a(b + c)$
10. $x(2y - z)$
11. $(x + y)z$
12. $3a + 2(a - b)$
13. $3(a - 2b) - 5a$
14. $-3(1 - x) - 2(x - 1)$
15. $4(a + 2) - 3(2 - a) - 5$
16. $x(x + 1) - x$
17. $x(2x + 3) - x^2$
18. $5x^2 + x(3x - 4)$

19. $5(a + 2b) - 3(a - 2b)$
20. $\frac{1}{2}(4a - 6b) - \frac{1}{4}(4a + 8b)$
21. $x(2x - 3y) - y(3x - 2y)$
22. $a(a - b) + b(2a - 3b)$
23. $b(3b - a) - a(3a + 2b)$
24. $x(2x - 3) - 4(3 - x)$
25. $2x(x - y) - 3y(x + y)$
26. $3x(2x + 3y) + 2x(3x - 2y)$
27. $4x(x + 2y) + 5(x^2 - 2xy)$
28. $a(a - b + c) - c(a - b - c)$
29. $a(b + c) - b(c - a) - c(a - b)$
30. $3a(3x - y) + 2a(x + y)$
31. $2x(3x - 2y) - 3y(4x + y) - 2(x^2 - 2y^2)$
32. $3c^2 - 2c(c - 2d) - 3d(c - d)$
33. $(a - 2b)b - (2a + b)a + 2(a^2 + b^2)$
34. $a^2(b - c) + b^2(c - a) + c(a^2 - b^2)$

Complete the following:

35. $4a + 8 = 4(\qquad)$
36. $3x - 18 = 3(\qquad)$
37. $x + 2y + 2z = x + 2(\qquad)$
38. $3a - 3b + c = 3(\qquad) + c$
39. $ab + ac = -a(\qquad)$
40. $x^2 + 3x - y = x(\qquad) - y$
41. $9a^2 - 3a = -3a(\qquad)$
42. $x^2 + 2x + a = x(\qquad) + a$
43. $3x + 3y + ax + ay = 3(\qquad) + a(\qquad)$
44. $ma - mb + na - nb = m(\qquad) + n(\qquad)$
45. $3x^3 - 2x^2 + 15x - 10 = x^2(\qquad) + 5($

Using brackets, write the following statements and simplify them:

46. Subtract x from y and multiply the result by a.

47. From three times x subtract the sum of y and x.

48. Take b times the sum of a and b from seven times the product of a and b.

CHAPTER 7

THE FOUR RULES WITH DIRECTED NUMBERS

Addition

Example: Add $3a + 4b - 2c$, $3c - 2b - 4a$, $-3b + 5c$.

Method I

$$
\begin{aligned}
(3a + 4b - 2c) + (3c - 2b - 4a) + (-3b + 5c) \\
&= 3a + 4b - 2c + 3c - 2b - 4a - 3b + 5c \\
&= 3a - 4a + 4b - 2b - 3b - 2c + 3c + 5c \\
&= -a - b + 6c
\end{aligned}
$$

Method II

$$
\begin{array}{rrr}
3a & + 4b & - 2c \\
-4a & - 2b & + 3c \\
 & - 3b & + 5c \\
\hline
-a & - b & + 6c \\
\hline
\end{array}
$$

Arrange the like terms in three columns as shown and add the terms in each column. Note that $+3c$ is written in the second line, *not* $3c$.

Exercises 19

Add:

1. $+5x$, $-8x$
2. $-5x$, $+6x$
3. $-2a^2$, $-3a^2$
4. $+ab$, $-3ab$
5. $-5abc$, $+5abc$
6. $\frac{1}{2}a$, $-\frac{1}{4}a$
7. $-\frac{2}{3}a^2$, $+\frac{1}{6}a^2$
8. $-5a + 6a - 7a + 3a$
9. $4x^2 - 7x^2 + 2x^2 - 5x^2 + 7x^2$

10. $+3a^2 - 5a^2 - 2a^2 + 10a^2 + a^2$
11. $-2ab - 7ab + 3ab - 4ab + ab$
12. $-1{\cdot}25x - 0{\cdot}75x - 0{\cdot}25x + 3x$
13. $\frac{1}{2}x^2 - \frac{1}{4}x^2 + 2x^2 - \frac{3}{4}x^2$
14. $\frac{1}{3}a - \frac{1}{4}a + a - \frac{1}{12}a$

15.
$$\begin{array}{r} 3a - 4b \\ -a + 2b \\ \hline \end{array}$$

16.
$$\begin{array}{r} -3x + 2y \\ -5x - 5y \\ \hline \end{array}$$

17.
$$\begin{array}{r} 4p - 7q \\ -3p + 9q \\ \hline \end{array}$$

18.
$$\begin{array}{r} 5x^2 - 3x \\ -6x^2 + 2x \\ 2x^2 + x \\ \hline \end{array}$$

19.
$$\begin{array}{r} -3a - 5b \\ 2a + 8b \\ -a - 2b \\ \hline \end{array}$$

20.
$$\begin{array}{r} 6a - 3b + 4c \\ -5a + b - 2c \\ \hline \end{array}$$

21.
$$\begin{array}{r} 3x + 2y - 5z \\ -4x + 3y - z \\ \hline \end{array}$$

22.
$$\begin{array}{r} 5x^2 - 3x + 2 \\ 2x^2 + 7x - 3 \\ \hline \end{array}$$

23.
$$\begin{array}{r} 4x^2 - 3x + 1 \\ -3x^2 + 2x - 7 \\ -x^2 - 5x + 6 \\ \hline \end{array}$$

24.
$$\begin{array}{r} 3x^2 + 2xy - y^2 \\ - 3xy + 2y^2 \\ -x^2 + 5xy \qquad \\ \hline \end{array}$$

25. $3a + 2b$ and $5b - 4a$
26. $5x - 4y$ and $3x + 2y$
27. $-2a + 4b$ and $3a - 2b$
28. $3(2x - 1)$ and $2(1 - 3x)$
29. $x(y - x)$ and $-y(y - x)$
30. $a + b - c,\ a - b + c,\ c - b - a$
31. $4a - 3b - 5c,\ -2a + b + 3c,\ a - b - c$
32. $a - (b + c),\ b - (2a - c),\ c - (b - 2a)$
33. $2(3a - b - c),\ 3(a - 2b - c),\ 4(2a - b + 3c)$
34. $2(x^2 - x + 2),\ -3(2x^2 - 3x + 1),\ 5 - 3x^2$
35. $3x^2 - 6 + 3x,\ -5 + 2x,\ 3 - 7x^2$
36. $a(a - b + c),\ b(a - c),\ c(b - a)$

37. $x(x-2y)+3y^2$, $3x^2-y(2y-x)$, $-2x^2-y^2$
38. $2b(c-a)$, $a(a+b)-c^2$, $-c(2b-c)$

Subtraction

Example: Subtract $3a^2-2a-7$ from a^2-3a-5.

Method I

$$\begin{aligned}(a^2-3a-5)-(3a^2-2a-7) &= a^2-3a-5-3a^2+2a+7\\ &= a^2-3a^2-3a+2a-5+7\\ &= -2a^2-a+2\end{aligned}$$

Method II

$$\begin{array}{r} a^2-3a-5\\ 3a^2-2a-7\\ \hline -2a^2-a+2 \end{array}$$

Arrange like terms in the same column; **mentally change the sign of each term in the lower line and add.**

Exercises 20

Subtract:

1. $5a$ from $7a$
2. $6a$ from $3a$
3. $2x$ from $-5x$
4. $-3x$ from $4x$
5. $-x$ from $-2x$
6. $\frac{1}{2}a^2$ from $\frac{1}{4}a^2$
7. $-ab$ from 0
8. $-3y^2$ from $-3y^2$
9. $\frac{5}{6}x^2$ from $-\frac{1}{3}x^2$

10. $\begin{array}{r}3a-2b\\ -a+4b\\ \hline\end{array}$

11. $\begin{array}{r}-5x+3y\\ -2x-7y\\ \hline\end{array}$

12. $\begin{array}{r}2a^2\\ -5a^2-3b^2\\ \hline\end{array}$

13. $\begin{array}{r}2a-4b-5c\\ 3a-b-c\\ \hline\end{array}$

14. $\begin{array}{r}x^2-3x+2\\ -3x^2+5x-8\\ \hline\end{array}$

15. $\begin{array}{r}-3ab+2bc-ca\\ -ab-3bc+ac\\ \hline\end{array}$

16. From $4x - 5y$ take $2x - 3y$.
17. From $-3a + 4b$ take $5b - 2a$.
18. From 0 take $2a - 5b$.
19. From $a + b - c$ take $c - a - b$.
20. Take $2x^2 + 5x$ from $7 - 3x + x^2$.
21. Take $2ab - b^2$ from $b^2 - ab + a^2$.
22. Take $-3x + 2y - z$ from $4x - 5y + 2z$.
23. Take $\frac{1}{2}x - \frac{1}{4}y + \frac{1}{8}z$ from $x - \frac{1}{2}y + \frac{1}{4}z$.
24. Take $5x^2 - 3xy + 2y^2$ from $2x^2 - 4xy + 8y^2$.
25. Take $x^3 - 2x^2 + 3x$ from $5x^2 - 7x + 2$.
26. Subtract the sum of $3a + 4b$ and $a - 2b$ from 0.
27. Add $2x^3 - 3x$ and $4 + 7x - x^2$ and subtract the result from $4x + 2x^2 + 4$.
28. Take the sum of $a + b - c$ and $b + c - a$ from the sum of $c + a - b$ and $a + b + c$.
29. Subtract $x(3x - 2y)$ from 0, and add the result to the product of $3x$ and $(x - y)$.
30. From $2x^2(x^2 - 2x + 3)$ take $3x^2(x - 2x^2 + 2)$.

Simplify:

31. $3(2a - 3) - 2(3a - 2) - 4(a - 3)$
32. $x(x - y) - 2x(3y - x) + 4(\frac{1}{2}xy - \frac{1}{4}x^2)$
33. $5ab - 3a(2a - 5b + c) + 6a(c + a)$
34. $\frac{1}{2}(4a + 8b) - \frac{1}{4}(6b - 2a) + 6\left(\frac{a}{3} - \frac{b}{6}\right)$
35. $2x(3x - 2y) - 3y(2x + y) - 3(x^2 - 2y^2)$

Multiplication

Example: $(3xy - 2x + 5) \times -2x$.

Each of the three terms $+3xy$, $-2x$ and $+5$ must be multiplied by $-2x$.

$$\therefore \text{ Expression} = -6x^2y + 4x^2 - 10x$$

Exercises 21

Simplify:

1. $2a \times -3$
2. $-5a \times -2$
3. $-3a \times 2b$
4. $4x \times -y$
5. $-2x \times -2x$
6. $2a^2 \times 3a$
7. $(-3a)^2$
8. $(2t)^2$
9. $2(-m)^2$
10. $(-2m)^2$
11. $a \times -b \times -c$
12. $-x \times -2y \times -z$
13. $-2a \times +2a \times +2a$
14. $(-3x)^3$
15. $(-a)^2 \times a^3$
16. $(-2a)^3 \times a^2$
17. $2(-m)^3 \times m^2$
18. $4x \times -xy$
19. $-3a \times -2ab$
20. $2x^2 \times 3xy$
21. $-3x^2y^2 \times -2xy$
22. $2xy^2 \times -5x^2y$
23. $-xy \times -yz \times zx$
24. $2ab \times -3a^2b^2 \times a^3b$
25. $4x^2y^2 \times \frac{1}{8}xy \times 2y$

26. Write down the squares of:

$$2a, -3b, -4ab, 3ab^2$$

27. Write down the cubes of:

$$-2x, 3y, -5ab, -\frac{2a}{3b}$$

Multiply:

28. $4a - 3$ by -2
29. $-a + b$ by -1
30. $2x + 3y$ by x
31. $3a - 2b$ by $-b$
32. $3x^2 - 1$ by -3
33. $4 - 5x^2$ by $-2x$
34. $x - y - z$ by $-x$
35. $3x^2 - 4x + 5$ by $2x$
36. $x^2 - 3xy$ by $-2xy$
37. $2x^2 - 3x + 1$ by $-3x$
38. $a^2 + ab + b^2$ by $-a$
39. $-2x + 1 - \frac{1}{2x}$ by $+4x$
40. $1 - x + 3x^2 - 2x^3$ by $-x$
41. $a - b + c$ by $-ab$
42. $+3a^2b - 2bc^2$ by $-bc$
43. $-2x^2 - 3x + 2$ by x^2
44. $1 - x + 2x^2 - 3x^3$ by $-2x^2$
45. $a^4 + 2a^3b - a^2$ by $-\frac{1}{a^2}$

Division

Example: $(6x^3y^2 - 4xy^2 - 8y) \div -2y$.

Each of the three terms $+6x^3y^2$, $-4xy^2$ and $-8y$ must be divided by $-2y$.

$$\frac{+6x^3y^2}{-2y} = -3x^3y, \quad \frac{-4xy^2}{-2y} = 2xy, \quad \frac{-8y}{-2y} = +4$$

$$\therefore \frac{6x^3y^2 - 4xy^2 - 8y}{-2y} = -3x^3y + 2xy + 4$$

Exercises 22

Simplify:

1. $8a \div -4$
2. $-6a \div 2$
3. $-10x^2 \div -5$
4. $+12ab \div +4a$
5. $6x \div -3x$
6. $-4xy \div -4xy$
7. $-6ab^2 \div +3a$
8. $+8xy^2 \div -2y^2$
9. $-9x^4 \div -3x^2$
10. $12x^4y^2z \div -x^2yz$
11. $-14x^3y^2 \div +2xy$
12. $30a^2b^2c^2 \div -6abc^2$

Divide:

13. $6a - 12$ by 2
14. $8x - 4$ by -4
15. $-10x^2 + 15$ by -5
16. $9a^2 - 6$ by 3
17. $ab - ac$ by a
18. $6h^2 + 8h$ by $-2h$
19. $4ab - 2a$ by $-2a$
20. $abc - ac + ac^2$ by ac
21. $k\,\underline{|\,ak - bk}$
22. $-a\,\underline{|\,2ax - 4ay}$
23. $t\,\underline{|\,t^3 + 3t^2 - t}$
24. $-5\,\underline{|\,10 + 5a - 20a^2}$
25. $3x\,\underline{|\,3x - 9x^2 - 15x^3}$
26. $-3b\,\underline{|\,9bx - 6by - 3bz}$
27. $bc\,\underline{|\,a^2b^2c - 2abc - 3b^2c^3}$
28. $ab\,\underline{|\,-a^3b^3 + ab - a^2b^2}$
29. $x^2\,\underline{|\,x^5 - 3x^4 + 2x^2}$
30. $-2a^2\,\underline{|\,-4a^4 + 2a^3 - 6a^2}$

Simplify:

31. $\dfrac{6t^2 + 8t}{2t}$

32. $\dfrac{4h - 8h^2 + 12h^3}{-4h}$

33. $\dfrac{-6x^2y^3 + 2y^2}{-2y^2}$

34. $\dfrac{15a^3b^2 - 12a^2b^3}{3ab}$

35. $\dfrac{6a^2b^2 - 12a^3b}{-4a^2b}$

36. $\dfrac{a^2b^2c^2 + 2abc^2 - 3a^2b^2c}{-abc}$

37. $3a^2 \times (2a)^3 \div 12a^4$

38. $(-2xy)^2 \times -3(x^2y)^2 \div -(x^2y)^2$

39. $\dfrac{x^4}{-x^2} - \dfrac{4x^3 - 2x^2}{2x^2} + \dfrac{3x^5 - 6x^3}{3x^3}$

40. $\dfrac{x^2y - xy^2}{xy} - \dfrac{3x^2}{x} + \dfrac{-12y^3}{4y^2}$

Chapter 8

SUBSTITUTION WITH DIRECTED NUMBERS

Example: If $a = +3$, $b = -2$, $c = +4$, find the value of:

(i) $(3a + 2b)^2$; (ii) $2b^3 + 3ab - \frac{c}{b}$.

When $a = +3$, $b = -2$, $c = +4$.

$$\begin{aligned}\text{(i) } (3a + 2b)^2 &= [3(+3) + 2(-2)]^2 \\ &= [+9 - 4]^2 \\ &= [+5]^2 \\ &= 25\end{aligned}$$

$$\begin{aligned}\text{(ii) } 2b^3 + 3ab - \frac{c}{b} &= 2(-2)^3 + 3(+3)(-2) - \frac{+4}{-2} \\ &= 2(-8) - 18 + 2 \\ &= -16 - 18 + 2 \\ &= -32\end{aligned}$$

Exercises 23

When $a = +2$, $b = +3$ find the value of:

1. $2a + 3b$
2. $(a + b)^2$
3. $a^2 + b^2$
4. $(a - b)^2$
5. $a^2 - b^2$
6. $3a - 2b$
7. $6ab$
8. $a^2 - 3ab$
9. $2a^2 - 5ab + 4$
10. $\dfrac{a^2 + b^2}{b - a}$
11. $(+2a)^3$
12. $(-2a)^3$
13. $2(-a)^3$
14. $2a^3$

When $a = -1$, $b = +2$, $c = -3$ find the value of:

15. $3b - 2a$
16. $4a - 2b + c$
17. $ab + bc + ca$
18. $2bc - 3ac$
19. $a^2 + b^2 + c^2$
20. $3(-a)^2 + (2b)^2$
21. $(-3a)^2 - 2(-2b)^3$
22. $3abc$
23. $\dfrac{ac}{b}$
24. $\dfrac{a-b}{b-c}$
25. $3a^2 - 4b^2 + c^2$
26. $3a(b + c)$
27. $(a - b)(b - c)$
28. $a^2 - 2bc + c^2$
29. $a(b - c) - c(a + b)$
30. $\dfrac{a^3 - b^3}{a^2 + ab + b^2}$
31. $\dfrac{1}{a} - \dfrac{1}{c}$

When $a = -2$, $b = -3$, $c = +1$, $d = 0$ find the value of:

32. $2b - 3a$
33. $3c - 2b + a$
34. $abcd$
35. $ab + bc + cd + da$
36. $(2a - 3b)^2$
37. $(3b - 2c)^2$
38. $3b^2 - 2c^2$
39. $\dfrac{cd}{a+b}$
40. $3a^2 + 5ab - b^2$
41. $\dfrac{a}{b} + \dfrac{b}{c}$
42. $a(b + c) - b(c - d)$
43. $\dfrac{b^2 - c^2}{b - c} + \dfrac{b^2 - a^2}{b - a}$
44. $a^3 + b^3 + c^3 + 3abc$
45. $a^2 + b^2 - 2a + 2b$
46. $\dfrac{a^3 + b^3}{a - b}$

Exercises 24

Prove that:

1. $3x^2 + 6x - 9 = 0$, when $x = -3$.
2. $(x - 3)^2 = 7(x - 3) - 6$, when $x = 9$.
3. $3x^2 + 14x - 17 = 2x - 2$, when $x = -5$.
4. $10x(x + 1) - 9(x + 2) = 24$, when $x = 2$.
5. $\dfrac{1}{x-2} - \dfrac{4}{x+1} + 3\frac{1}{3} = 0$, when $x = \frac{1}{2}$.

6. When $a = 3$, $b = -1$, find the value of $\dfrac{a^2 + b^2}{a^2 - ab + b^2}$.

7. When $x = 2$, $y = -3$, find the value of $\dfrac{(x - y)^3}{x^2 + 3xy + 6y^2}$.

8. If $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{f}$, find f when $u = 36$, $v = -12$.

9. If $2x - 3y = 5$, find the value of x when $y = 2$, and the value of y, when $x = -2$.

10. If $a = -3$, $b = \frac{2}{3}$, find the values of:

(i) $a^3(b - 2)$ (ii) $b^2 + \dfrac{5}{a} + 1$.

11. If $x = -\frac{3}{2}$, prove that $\dfrac{2x + 3}{x - 3} + \dfrac{2x + 3}{2x + 1} = 0$.

12. If $x = -2$, prove that $x^3 - 3x^2 - 4x + 12 = 0$.

13. Complete the following table:

x	-2	-1	0	1	2
$2x^2$	8				
$+5x$	-10				
$+3$	$+3$	$+3$	$+3$	$+3$	$+3$
$2x^2 + 5x + 3$	1				

Draw up tables similar to the above, and find the value of the following expressions from $x = -2$ to $x = +2$:

14. $x^2 + 5x + 6$

15. $-x^2 + 3x - 4$

16. $2x^2 + 4x - 5$

17. $1 - 4x - 3x^2$

18. $x^3 - 2x^2 - 3x + 7$

19. Find the values of $x^3 - 8x$ for the following values of x: $-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$.

20. Find the values of $(2x + 1)(3x - 5)$ for the following values of x: $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}$.

21. Find the values of $\dfrac{3(x^2 - 1)}{x(x - 2)}$ for the following values of x: $-2, -1, 1, 3, 4$.

22. When $x = -3$, $y = -1$, $a = 2$, find the value of

$$\frac{x^3 - y^3}{x + y} - \frac{x^2 - a^2}{x - a}$$

23. If $x = 2y - 7$, and $a = 3x + 2$, find the value of a when $y = 4$.

24. If $x = \dfrac{8 - 3y}{4y + 1}$, and $a = \dfrac{x}{2} - 4$, find the value of a when $y = -4$.

25. If $a = x + 3y$ and $b = 3x + y$, find the value of $a^2 + ab$ when $x = -3$, $y = +4$.

REVISION PAPERS 6–10

Paper 6

1. When $x = 2, y = -3$, find the value of:

(i) $3x + 2y$ (iii) $(2x + y)^3$
(ii) $2(x - y)^2$ (iv) $(3x)^2 - 3y^2$

2. Simplify $(2a - 3b + c) - (4a + 3b - 4c) - 2(a - b)$.

3. Complete the following:

(i) $2a + 4b = 2(\qquad)$
(ii) $xy - xz = x(\qquad)$
(iii) $-3a + 6 = -3(\qquad)$

4. Simplify:

(i) $\frac{5a}{3b^2} \div \sqrt{\frac{100a^2}{9b^2}}$ (ii) $\frac{ab^2}{-b} \times \frac{-b}{a^3} \div \frac{-b}{a}$

5. If $3y = 4x + k$ and $y = 2$ when $x = -2$, find the value of k. Use this value for k in the equation to find the value of y when $x = -5$.

6. A man bought $(x - y)$ articles at 6*s*. each, and $(x + y)$ articles at 5*s*. each. He sold them all at 8*s*. each. Find his profit in pounds.

Paper 7

1. Add $3x^2 - 4x - 5$, $-4x^2 + x + 3$, $11 - 5x + 3x^2$.

2. Simplify $3(5 - 2x) - 2(6 - 5x) - (7 + x)$.
What is the value of the answer when $x = -3$?

3. A man buys a articles at $(b - 3)$ shillings each. All but 4 are sold at b shillings each, the 4 being sold at half this price each. Find his profit in shillings.

4. If $a = -2$, $b = -1$, find the value of:

(i) $a^3 - a^2b - ab^2 + b^3$ (ii) $\dfrac{a^2 + ab + b^2}{a - b}$

5. Simplify:

(i) $3ab^2 \times -2a^2b \times ab$
(ii) $(-6a)^2 \div -2a^2$
(iii) $(-2a + 3b - c) \times -ab$
(iv) $\dfrac{8x^3y - 4xy^3 + 2xy}{-2xy}$

6. If $y = \dfrac{6}{2 + x}$ and $z = 3y(2 - x)$, find z when $x = -4$.

Paper 8

1. Simplify $3(2 - 3x + 5x^2) - 2(x^2 - 2x - 1) - (8 + 5x)$.

2. When $a = -2$, $b = -1$, find the value of:

(i) $2(3a - 4b)^3$ (ii) $2(3a^3 - 4b^3)$.

3. Simplify:

(i) $(-2x)^3 \times 3x^2$
(ii) $-2x^3 \times 3x^2 \times -x$
(iii) $(1 - 3x - 4x^2) \times -2x^2$
(iv) $(x^4 - 2x^3y + x^2y^2) \times \dfrac{1}{x^2}$

4. Two-thirds of a certain number exceeds 6 by 4. Find the number.

5. If $x = a^2 - 2b^2$ and $y = (2a)^2 - b^2$, find the value of $\dfrac{x + y}{x - y}$ when $a = 2$, $b = -3$.

6. A train due to arrive at x min. before noon was delayed and arrived at x min. past b o'clock. Find an expression for the number of minutes it was late.

If it arrived between 2 and 3 o'clock, 140 minutes late, find exactly when it was due to arrive.

Paper 9

1. By how much does $3x^2$ exceed the sum of $2 - 5x + x^2$ and $2x^2 + 3x - 1$?

2. Simplify $b(a + b - c) - c(b - c - a) - a(c + b)$.

3. Simplify:

(i) $(-2a)^2 \div -2a$

(ii) $-2a^4 \div -2a$

(iii) $(-3a)^2 \times 3a^2 \div (-a)^3$

(iv) $(2a^2b - ab^2 - 3ab) \div -ab$

4. When $x = -2$, $y = -3$, $\dfrac{3x + 2y}{2x - 3y} - \dfrac{x - y}{x + y} = \dfrac{a}{10}$.

Find the value of a.

5. A silver collection consisted of x half-crowns, $2x$ florins and $3x$ shillings. What was the value in sixpences?

If the value was £9 10*s*., how many half-crowns were there?

6. A man bought a lb. tea at x shillings per lb., and also x lb. tea at a shillings per lb. He mixed them and sold the mixture at 8 shillings per lb. Find his gain per lb. If $a = 7$, $x = 5$, find the gain per lb.

Paper 10

1. If $s = ut + \frac{1}{2}ft^2$, find s if $u = 10$, $f = -3$, $t = 4$. Then find v if $v^2 = u^2 + 2fs$.

2. If $a = 2x - 3y$, $b = 3x - 2y$, find the value of $4a - 3b - 2(x - y)$. Then find the value of the answer if $x = y = -2$.

3. Simplify $\dfrac{x^2}{-x} + (-x)^2 - \dfrac{x^3 + x^4}{x^2}$.

Find the value of the answer when $x = -3$.

4. Simplify:

$3a(2a - 5b) - 2b(4b - a) - 2(a^2 - 8ab + 3b^2)$

5. (i) What number is less than x by as much as y is greater than x?

(ii) If b is greater than c by 4, and c is greater than the square of a by 3, find a when $b = 16$.

6. The side of a square is $2x$ yd. long. A rectangle with the same perimeter as the square is $10x$ ft. long. What is its breadth? What is its area in sq. ft.?

By how much does the area of the square exceed the area of the rectangle? Answer in sq. ft.

Chapter 9

H.C.F.—L.C.M.—FRACTIONS I

Degree

The term xy has *two letter factors*. It is said to be a term of the *second degree* or a term of *two dimensions*. Similarly x^2, i.e. xx, is a term of the second degree or of two dimensions. x^3, $3x^2y$, $5xyz$ are each said to be of the third degree, or of three dimensions, since each of them has three letter factors.

The term $7x^3y^2z$ is thus a term of the sixth degree or of six dimensions.

When adding or subtracting algebraic expressions such as $3x^2 - 5 + 7x - 4x^3$ and $6x + 2x^3 - 5x^2 - 7$, we found it convenient to arrange the terms in these two expressions in one of the following two orders:

1.
$$\begin{array}{l} -4x^3 + 3x^2 + 7x - 5 \\ +2x^3 - 5x^2 + 6x - 7 \end{array}$$

where the terms are arranged in *descending order* of the powers of x.

2.
$$\begin{array}{l} -5 + 7x + 3x^2 - 4x^3 \\ -7 + 6x - 5x^2 + 2x^3 \end{array}$$

where the terms are arranged in *ascending order* of the powers of x.

In later work it will be found useful to arrange the terms in algebraic expressions either in ascending order or in descending order of the powers of one of the letters in use.

Exercises 25

1. State the degree of each of the following terms: x, $3x^2$, $7x$, $2x^3$, ab, $2c$, $3xyz$, $5x^2y$, xy^3, $2x^2y^3$.

2. Arrange the following expressions in descending order:

(i) $3x + 2 + 5x^2$
(ii) $3 - 7x^2 + 4x$
(iii) $8x^2 - x + 2x^3 - 4$
(iv) $3x^3 - 7 + 5x^4$
(v) $a^4 - 2a^2 + 5a^3 - 3a$
(vi) $2b - 3b^2 + 7b^3 - 8$

3. Arrange the following expressions in ascending order;

(i) $3a^2 + 7 - 2a$
(ii) $5x^3 - 3 + 2x^2 - 7x$
(iii) $x^2 + 2x^4 - 3x + 5x^3$
(iv) $3x^3 - 5x^5 + 2x^2 - 7x$
(v) $7b + 1 - 2b^4 + 3b^2$
(vi) $6x(1 - x^2) - 2(3x^4 - 5)$

Highest Common Factor (H.C.F.)

Consider the expressions x^2y^2 and xy^3:

$$x^2y^2 = xxyy$$
$$xy^3 = xyyy$$

There are 5 factors common to both, namely x, y, xy, y^2, xy^2. Of these 5 factors x and y are each of the first degree, xy and y^2 are each of the second degree, while xy^2 is of the third degree. The common factor of the highest degree is therefore xy^2; which is thus said to be the Highest Common Factor (*H.C.F.*) of the expressions x^2y^2 and xy^3.

Example 1: Find the H.C F. of ab^2c, bc^2d, $abcd^2$.

The only letters that occur in all three expressions are b and c, and in each case the first power is the only one common to all three.

$\therefore$ The H.C.F. is bc.

Example 2: Find the H.C.F. of $8x^2y^2$, $12xy^3$, $20x^2y^3$.

The H.C.F. of 8, 12 and 20 is 4
,, x^2, x, x^2 is x
,, y^2, y^3, y^3 is y^2

$\therefore$ The H.C.F. of the three expressions $= 4xy^2$.

Example 3: Find the H.C.F. of $x(x+1)$, $(x+1)^2$, $x^2(x+1)$.

$x(x+1)$ contains two factors x and $(x+1)$
$(x+1)^2$,, ,, ,, $(x+1)$ and $(x+1)$
$x^2(x+1)$,, three ,, x, x and $(x+1)$

The only factor common to all three is $(x+1)$, which is therefore the H.C.F.

Exercises 26

Find the H.C.F. of:

1. a, a^2
2. b^3, b^2
3. $6a, 9a$
4. $8x, 12x$
5. xy, xz
6. ab, bc
7. ab^2, a^2b
8. a^2, a^4
9. a, a^2, a^3
10. $5x, 10x, 15x$
11. $2x, 4x^2, 6x^3$
12. $4ab, 6a^2b^2$
13. a^2b, ab^3
14. ab^3, a^3b^2
15. a^2b^2c, a^2b^3c
16. $9a^2b^3, 12a^3b^2$
17. $15x^4y^3, 18x^5y^2$
18. $12, 3a, 6ab$
19. $14x^2y^2, 21x^3y$
20. $8p^2qr, 16pr$
21. $2x^2, 6x, 16x^3$
22. $9a^2x^3, 15ax^2$
23. a^4, a^3, a^2
24. abc, bcd, cda
25. $x^2y, 2xy, 3y^2$
26. $8a^2, 12ab, 20ab^2$
27. $6x^4, 9x^3, 12x^2$
28. $2ab, 6bc, 8bcd, 10abc$
29. $6a^2b, 8a^2b^2, 12ab^2, 10b^3$
30. $x^2y^3, x^3y^2, x^4y^3, x^3y^4$
31. $x^2(x+2), x^3, x(x+2)^2$
32. $x(x+1)^2, x^2(x+1), x^3(x+1)(x+2)$

Fractions

In Arithmetic, to simplify a fraction such as $\frac{30}{48}$ we divide both numerator and denominator by the H.C.F. of 30 and 48, i.e. 6.

$$\therefore \frac{30}{48} = \frac{30 \div 6}{48 \div 6} = \frac{5}{8}$$

This is usually shown thus:

$$\frac{\overset{5}{\cancel{30}}}{\underset{8}{\cancel{48}}} = \frac{5}{8}$$

Similarly, in Algebra to simplify a fraction such as $\frac{ab^3}{a^2b^2c^2}$ we divide both numerator and denominator by the H.C.F. of ab^3 and $a^2b^2c^2$, i.e. ab^2.

$$\therefore \frac{ab^3}{a^2b^2c^2} = \frac{ab^3 \div ab^2}{a^2b^2c^2 \div ao^2} = \frac{b}{ac^2}$$

This is usually shown thus:

$$\frac{\overset{1}{\cancel{a}}\overset{b}{\cancel{b^3}}}{\underset{a}{\cancel{a^2}}\underset{1}{\cancel{b^2}}c^2} = \frac{1 \times b}{a \times 1 \times c^2} = \frac{b}{ac^2}$$

With experience the intermediate step $\frac{1 \times b}{a \times 1 \times c^2}$ may be omitted.

Example: Simplify $\frac{4ax^2}{b} \times \frac{b^3}{9xy^2} \div \frac{-2b}{3y}$.

Expression $= \frac{\overset{2\ x}{\cancel{4ax^2}}}{\underset{1}{\cancel{b}}} \times \frac{\overset{b}{\cancel{b^3}}}{\underset{3\ 1\ y}{\cancel{9xy^2}}} \times \frac{\overset{1\ 1}{\cancel{3y}}}{\underset{1\ 1}{\cancel{-2b}}}$

$$= \frac{2 \times a \times x \times b \times 1 \times 1}{1 \times 3 \times 1 \times y \times -1 \times 1}$$

$$= -\frac{2axb}{3y}$$

Exercises 27

Simplify:

1. $\dfrac{2a}{4b}$
2. $\dfrac{6a}{3}$
3. $\dfrac{-5x}{5}$
4. $\dfrac{6b}{-3a}$
5. $\dfrac{ax}{bx}$
6. $\dfrac{y}{3y}$
7. $\dfrac{-ab}{-ac}$
8. $\dfrac{2xy}{5xy}$
9. $\dfrac{y}{y}$
10. $\dfrac{2a^2}{a}$
11. $\dfrac{x^2}{-xy}$
12. $\dfrac{5ab}{b}$
13. $\dfrac{x^4}{-x^2}$
14. $\dfrac{-5x^3}{x}$
15. $\dfrac{-x^2}{3x^4}$
16. $\dfrac{12x^3y}{3xy}$
17. $\dfrac{4a}{-20a^3}$
18. $\dfrac{a}{5a^3}$
19. $\dfrac{3x}{21x^2}$
20. $\dfrac{-xy}{xy}$
21. $\dfrac{2(-2x)}{6x}$
22. $\dfrac{(-2x)^2}{6x}$
23. $\dfrac{5a^2}{-5a^3}$
24. $\dfrac{3a^2b^2}{12a^3b}$
25. $\dfrac{(xy)^3}{xy}$
26. $\dfrac{(-2ab)^2}{2ab^2}$
27. $\dfrac{xy + xyz}{xy}$
28. $\dfrac{-3a^2b^2c}{-abc}$
29. $\dfrac{-9a^2b^2c^3}{-3abc}$
30. $\dfrac{a^2b^2c^3}{(ab)^2}$
31. $\dfrac{5(x + y)}{5}$
32. $\dfrac{3x(a + b)}{-6x}$
33. $\dfrac{a + b}{-(a + b)}$
34. $\dfrac{(a^2b)^3}{a^3b^2}$

Exercises 28

1. Write down the reciprocals of:

$$a, \frac{1}{a}, 3a, -2a, \frac{a}{b^2}, \frac{1}{2a}, -\frac{b^2}{a^2}, 3\tfrac{1}{7}, \frac{5}{4x}$$

Simplify:

2. $x \times \frac{1}{x}$

3. $x^2 \times \frac{1}{x}$

4. $x \times -\frac{1}{x}$

5. $-x \times \frac{1}{x}$

6. $x^2 \times \frac{-1}{x}$

7. $\frac{a}{b} \times \frac{b}{a}$

8. $\frac{a^2}{b} \times \frac{b^2}{a}$

9. $\frac{-a^2}{b} \times \frac{-b^2}{a}$

10. $\frac{-a}{b} \times \frac{b^2}{a^2}$

11. $\frac{-x^2}{y} \times \frac{-y^2}{x^2}$

12. $\frac{a^2b \times bc^2}{abc}$

13. $\frac{(2m^3)(4m)}{(-2m)^3}$

14. $\frac{-6a^2}{9b^2} \times \frac{3b}{2ac}$

15. $6x \div \frac{1}{6x}$

16. $\frac{1}{\frac{1}{a}}$

17. $\frac{\frac{x}{a}}{x}$

18. $\frac{\frac{1}{2}a^2b^2}{2cd^2} \times \frac{3cd}{\frac{1}{4}ab}$

19. $\frac{8}{4x} \div \frac{2}{3x^2}$

20. $2a^2b^3 \div 4a^2b$

21. $8x \div -\frac{1}{2x}$

22. $\frac{-x^2}{y^2} \div -\frac{x}{y}$

23. $\frac{4a^2}{14b} \div \frac{2a}{7b^2}$

24. $\frac{ab}{c} \times \frac{bc}{a} \times \frac{ca}{b}$

25. $\frac{a}{b} \times \frac{b}{c} \div \frac{c}{a}$

26. $\frac{a^2b}{-b^2c^2} \times \frac{-c^3}{a} \div \frac{-c}{a}$

27. $\frac{(-x)^3}{y^3} \times \frac{y}{2x} \times \frac{-y}{x}$

28. $\frac{4}{a} \times \frac{1}{2b} \div \frac{2}{c}$

29. $\frac{25a^2b}{15b^2c} \times \frac{3b}{5a} \div \frac{4b}{a^2c}$

30. $\frac{(-2xy)^3}{-3x^2} \times \frac{3(-x)^2}{4y^2} \times \frac{-y}{6x^2}$

31. $\frac{6c}{15b} \times \frac{2ba^2}{3c^2} \div \frac{4a}{9bc}$

32. $\frac{\frac{2}{3}x^2y}{\frac{1}{6}yz} \times \frac{\frac{3}{4}z^2}{\frac{2}{5}y} \div \frac{\frac{5}{6}x}{\frac{1}{2}y^2}$

Lowest Common Multiple (L.C.M.)

Consider the expressions x^2y^2 and xy^3.

Any expression containing x^2y^2 as a factor is a multiple of x^2y^2, e.g. $x^2y^2 \times x$, i.e. x^3y^2, or $x^2y^2 \times z$, i.e. x^2y^2z, or $x^2y^2 \times 2y^4$, i.e. $2x^2y^6$, to mention only three.

Similarly, any expression containing xy^3 as a factor is a multiple of xy^3, e.g. xy^4, xy^3z or $2x^3y^3$ to mention only three.

A multiple common to both x^2y^2 and xy^3 must contain the factor x^2 and also the factor y^3. For example, x^2y^3, x^3y^3, x^3y^4, x^4y^3, etc., are all common multiples of x^2y^2 and xy^3. These multiples are of degree 5, 6, 7, 7 respectively. The common multiple of lowest degree is x^2y^3, which is said to be the Lowest Common Multiple (L.C.M.) of x^2y^2 and xy^3.

Example 1: Find the L.C M. of $18xy^2z$ and $12x^2y$.

The L C M. of 18 and 12 is 36
,, x and x^2 is x^2
,, y^2 and y is y^2

and the L.C.M. must contain the factor z because z is a factor of $18xy^2z$.

$\therefore$ The L.C.M. of $18xy^2z$ and $12x^2y$ is $36x^2y^2z$.

Example 2: Find the L.C M. of $x(x+1)$, $(x+1)^2$, x^2.

The required L.C.M. must contain the factors x and $(x+1)$, since it is to be a multiple of $x(x+1)$.

Since it is to be a multiple also of $(x+1)^2$, it must contain the factor $(x+1)^2$, and similarly, since it is to be a multiple of x^2, it must contain the factor x^2.

$\therefore$ The required L.C.M. is $x^2(x+1)^2$.

Example 3: Express the following fractions with lowest common denominators:

$$\frac{a}{b},\ \frac{3b}{5c},\ \frac{2c}{3a}$$

The L.C.M. of the three denominators, b, $5c$, $3a$ is $15abc$. Each fraction is now expressed with $15abc$ as its denominator. The denominator b of the first fraction must therefore be multiplied by $15ac$, and so, of course, must its numerator.

$$\therefore \frac{a}{b} = \frac{a \times 15ac}{b \times 15ac} = \frac{15a^2c}{15abc}$$

$$\text{Similarly,} \quad \frac{3b}{5c} = \frac{3b \times 3ab}{5c \times 3ab} = \frac{9ab^2}{15abc}$$

$$\frac{2c}{3a} = \frac{2c \times 5bc}{3a \times 5bc} = \frac{10bc^2}{15abc}$$

Exercises 29

1. State any 3 multiples of $2a$, $3ab$, $4abc$.
2. State any 3 multiples of a^2, $2a^3$, $6a^2b$.
3. State any 3 common multiples of x, x^2 and xy.
4. State any 3 common multiples of $2x^2$, $8xy^3$ and $6x^3y$.

Find the L.C.M. of:

5. a, $2a$
6. $2a$, $3a$
7. $2a$, $3b$
8. $4x$, $6y$
9. $6x$, $15y$
10. $9x$, $15x$
11. $2x$, $4x$, $6x$
12. $3a$, $6b$, $9c$
13. a, a^2
14. $2a$, $3a^2$
15. $6a^2$, $9a^4$
16. x^2, $3x$, $2x^3$
17. $3a$, $15ab$
18. $4a^2$, $6ab$
19. $12ab^2$, $9a^2b$
20. $4xy$, $10x^2y^2$
21. ab, bc, ca
22. $2ab$, $3bc$, $6ca$
23. a^2b, ab^2, ac^2
24. xy^3, x^2y^2, x^4
25. $4x^2y$, x^2y^3, $6xy^3$
26. $(x+1)$, $(x+2)$
27. x^2, x^3, $(x+1)^2$
28. $x^2(x+2)$, $(x+2)^3$, $x(x+2)^2$

Express with lowest common denominators:

29. $\frac{a}{3}, \frac{a}{5}$
30. $\frac{3}{a}, \frac{4}{ab}$
31. $\frac{2}{x}, \frac{3}{x^2}$
32. $a, \frac{1}{a}$
33. $\frac{3}{a}, \frac{5c}{ab}$
34. $\frac{a}{5}, \frac{b}{a}$

35. $\frac{3a}{2b}, \frac{5}{bc}$

36. $\frac{c}{4b^2}, \frac{a}{3bc}$

37. $\frac{2x}{3y^2}, \frac{5}{6x^2y^2}$

38. $\frac{1}{a}, \frac{2}{a^2}, \frac{3}{a^3}$

39. $\frac{a}{b}, \frac{b}{c}, \frac{a^2}{c^2}$

40. $1, \frac{3}{ab^2}, \frac{4}{abc}$

41. $\frac{a}{x}, \frac{b}{2y^2}, \frac{c}{(3xy)^2}$

42. $\frac{1}{x}, \frac{2}{x+1}$

43. $\frac{a}{x+1}, \frac{b}{(x+1)^2}$

44. $\frac{2}{x(x+1)}, \frac{3}{x^2(x+1)^2}, \frac{4}{x^3}$

Addition and Subtraction of Fractions

In Algebra, to add or subtract fractions, we proceed in the same way as we do in Arithmetic, i.e. we express the fractions with a common denominator, usually the lowest common denominator.

$$\frac{1}{3}+\frac{1}{4}$$

$$=\frac{1\times 4}{3\times 4}+\frac{1\times 3}{4\times 3}$$

$$=\frac{4}{3\times 4}+\frac{3}{4\times 3}$$

$$=\frac{4+3}{3\times 4}=\left[\frac{7}{12}\right]$$

$$\frac{1}{a}+\frac{1}{b}$$

$$=\frac{1\times b}{a\times b}+\frac{1\times a}{b\times a}$$

$$=\frac{b}{ab}+\frac{a}{ba}$$

$$=\frac{b+a}{ab}$$

$$\frac{2}{3}+\frac{4}{5}$$

$$=\frac{2\times 5}{3\times 5}+\frac{4\times 3}{5\times 3}$$

$$=\frac{2\times 5+4\times 3}{3\times 5}$$

$$=\left[\frac{10+12}{15}\right]$$

$$=\left[1\tfrac{7}{15}\right]$$

$$\frac{a}{b}+\frac{c}{d}$$

$$=\frac{a\times d}{b\times d}+\frac{c\times b}{d\times b}$$

$$=\frac{ad}{bd}+\frac{cb}{db}$$

$$=\frac{ad+cb}{bd}$$

Example: Simplify $\frac{2}{3x}-\frac{4}{y}+\frac{3}{x}$.

The lowest common denominator is $3xy$.

$$\frac{2}{3x}-\frac{4}{y}+\frac{3}{x}=\frac{2y}{3xy}-\frac{12x}{3xy}+\frac{9y}{3xy}$$
$$=\frac{2y-12x+9y}{3xy}$$
$$=\frac{11y-12x}{3xy}$$

Exercises 30

Complete the following:

1. $\frac{a}{4}=\frac{\quad}{8}=\frac{3a}{\quad}=\frac{\quad}{12a}=\frac{\quad}{4y}$
2. $\frac{3}{a}=\frac{12}{\quad}=\frac{9a}{\quad}=\frac{\quad}{a^2}=\frac{\quad}{2ax}$
3. $\frac{x}{y}=\frac{4x}{\quad}=\frac{\quad}{yz}=\frac{3x^2}{\quad}=\frac{\quad}{xy^2}$
4. $\frac{x}{3y}=\frac{2x}{\quad}=\frac{\quad}{3y^2}=\frac{5x^2}{\quad}=\frac{-2x}{\quad}$
5. $\frac{a}{b}=\frac{-a}{\quad}$
6. $\frac{a}{-b}=\frac{-a}{\quad}$
7. $\frac{-a}{b}=\frac{a}{\quad}$
8. $\frac{-a}{-b}=\frac{\quad}{b}$
9. $\frac{9a}{-24b}=\frac{3a}{\quad}$
10. $x=\frac{\quad}{x}$
11. $2a^2=\frac{\quad}{a^2}$
12. $\frac{1}{3x^2}=\frac{-x}{\quad}$
13. $\frac{4x}{6y}=\frac{-2x}{\quad}$
14. $\frac{-a}{6a^2b}=\frac{\quad}{-12a^2bc}$
15. $\frac{-1}{9x^2y}=\frac{\quad z^2}{\quad}$
16. $\frac{(-2a)^3}{16a^2b}=\frac{a^3}{\quad}$

Simplify:

17. $\frac{x}{3}+\frac{x}{5}$
18. $\frac{x}{3}-\frac{x}{4}$

19. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4}$

20. $\frac{x}{3} + \frac{x}{4} - \frac{x}{5}$

21. $\frac{2a}{3} - \frac{3a}{5}$

22. $\frac{1}{2x} + \frac{1}{3x}$

23. $\frac{2}{3x} - \frac{3}{4x}$

24. $\frac{3}{2a} - \frac{4}{5a}$

25. $1 + \frac{1}{a}$

26. $1 - \frac{1}{a}$

27. $\frac{1}{x} + \frac{1}{y}$

28. $\frac{1}{b} - \frac{1}{a}$

29. $\frac{a}{b} + 1$

30. $1 - \frac{a}{b}$

31. $\frac{2}{a} + \frac{3}{b}$

32. $\frac{5}{xy} - \frac{3}{y}$

33. $\frac{4}{x^2} - \frac{5}{x}$

34. $a - \frac{b}{a}$

35. $\frac{2}{a} + \frac{a}{2a^2}$

36. $\frac{2}{ab} - \frac{3}{a^2}$

37. $\frac{x}{y} - \frac{x^2}{y^2}$

38. $\frac{a^2}{b} + \frac{b^2}{a}$

39. $\frac{1}{a^2} - \frac{1}{a} + 1$

40. $\frac{3a}{4} - \frac{5a}{8} + \frac{a}{6}$

41. $\frac{a}{b} - \frac{a^2}{2b^2}$

42. $\frac{2}{x} + \frac{3}{x^2} - 1$

43. $\frac{a}{b} + \frac{b}{a} + 2$

44. $\frac{3}{ab} - \frac{2}{a} + \frac{1}{b}$

45. $\frac{3}{x^2y} + \frac{4}{xy^2} - 2$

46. $\frac{3}{x^3} - \frac{2}{x^2} - \frac{1}{x}$

47. $\frac{2}{a} - 3 + \frac{5}{2a}$

48. $\frac{5}{3y} + \frac{2}{y} - \frac{3}{2y}$

49. $\frac{4}{3a} - \frac{3}{4a} + \frac{2}{a}$

50. $\frac{2}{xy} - \frac{3}{x} + \frac{4}{y}$

51. $\frac{3}{2x} - \frac{5}{x} - \frac{2}{x}$

52. $\frac{1}{x} - \frac{1}{y} + \frac{1}{z}$

53. $\frac{1}{a^2} + \frac{3}{ab} \times \frac{b}{6}$

54. $x + \frac{1}{x} - \frac{3}{x^2}$

55. $\frac{3}{abc} - \frac{2}{bca} + \frac{1}{cab}$

56. $\frac{5}{xy} - \frac{2x}{y} + \frac{3y}{z}$

57. $\frac{x^2}{y^2} - \frac{3y}{2x} \times \frac{x^2}{y^2}$

58. $\frac{a-b}{4} + \frac{a+b}{3}$

59. $1 + \frac{a-b}{b}$

60. $\frac{a}{2a} + \frac{b}{3b^2} - \frac{4c}{12bc}$

61. $\dfrac{1}{1 + \frac{1}{b}}$

62. $\dfrac{1}{\frac{a}{b} - 1}$

63. $\dfrac{1}{\frac{1}{a} + \frac{1}{b}}$

64. $\dfrac{y}{y - \frac{1}{y}}$

65. $\dfrac{\frac{1}{b}}{\frac{a}{b} - \frac{b}{a}}$

CHAPTER 10

BRACKETS II

THE rules on pages 40 and 42 should be revised.

In Chapter 6 we have considered the use of single brackets in expressions such as $3a - 2(x - y)$. If we try to express with letters, signs and brackets, five times the excess of $3a$ over $2(x - y)$, we find that we require to use two sets of brackets, thus:

$$5(3a - 2(x - y))$$

To avoid confusion it is better to use two different forms of brackets and write the above expression thus:

$$5\{3a - 2(x - y)\}$$

Example 1: Simplify $2(5a - 2y) - \{3a - 2(x - y)\}$

$$\begin{aligned} \text{Expression} \quad &= 10a - 4y - \{3a - 2x + 2y\} \\ &= 10a - 4y - 3a + 2x - 2y \\ &= 7a + 2x - 6y \end{aligned}$$

The different forms of brackets are:

Single bracket	$(a - b)$
Double bracket	$\{a - b\}$
Square bracket	$[a - b]$
Vinculum	$\overline{a - b}$

Note that the line in the fraction $\frac{a - b}{3}$ or $\frac{3}{a + b}$ in addition to indicating division, serves as a vinculum.

$\dfrac{a-b}{3}$ is the same as $(a - b) \div 3$ or $\frac{1}{3}(a - b)$.

$\dfrac{3}{a+b}$ is the same as $3 \div (a + b)$.

When two or more forms of brackets occur, the best plan is to remove the **innermost first.**

Example 2: Simplify $3[2x - 4(2y - \overline{x - y})]$.

$$\begin{aligned} \text{Expression} &= 3[2x - 4(2y - x + y)] \\ &= 3[2x - 8y + 4x - 4y] \\ &= 6x - 24y + 12x - 12y \\ &= 18x - 36y \end{aligned}$$

Exercises 31

Simplify:

1. $5a + (3a - 2)$
2. $4a - (2a + 5)$
3. $a - \overline{a - b}$.
4. $a - b - \overline{a + b}$
5. $3(a + 2b) - 2(2a + b)$
6. $a(x - 2) + 3(a - x)$
7. $x(2x - 3) - 3x(5 - 2x)$
8. $x(y - z) - y(x + z)$
9. $a - a\left(1 + \dfrac{1}{a}\right)$
10. $x\left(1 + \dfrac{1}{x}\right) - x\left(1 - \dfrac{1}{x}\right)$
11. $2(a + 2b) + 3(2b - a) - 4(3a - b)$
12. $a(b - c) - b(c - a) - c(a - b)$
13. $5a + \{3a - (a - 2b)\}$
14. $(3x - 2y) - \{2x - (2x - y)\}$
15. $2\{(x - 2y) - 3\} - 3(3x - y)$
16. $2a(2a - b) - 2b[b - (a - b)]$
17. $3[2(x - \overline{y - 1})]$
18. $2t[3\overline{t - 1} - 2] - 6t^2$
19. $5x - 3[2x + (x - \overline{2x - y})]$.

Find the value of the answer when $x = -2$, $y = 1$.

20. $3(a^2 - b^2) - \dfrac{2a^2 - 4b^2}{2}$.

Find the value of the answer when $a = b = -1$.

21. $\dfrac{n}{2}[2a + \overline{n - 1}d]$.

What is tho value of the answer when $n = 10$, $a = 1$, $d = -2$?

22. $a\{a + \overline{b - a}\} - b\{b - \overline{a - b}\}$.

Find the value of the answer, when $a = 2$, $b = 3$.

Complete the following:

23. $x - y + z = x - (\qquad\qquad)$
24. $x + y - z = x + (\qquad\qquad)$
25. $5a^2 + 3a - 6 = 5a^2 + 3(\qquad\qquad)$
26. $3a^2 - 4a + 4 = 3a^2 - 4(\qquad\qquad)$
27. $ab + ac + 3b + 3c = a(\qquad\qquad) + 3(\qquad\qquad)$
28. $ab - ac - 3b + 3c = a(\qquad\qquad) - 3(\qquad\qquad)$
29. $xy + x - 2y + 2 = x(\qquad\qquad) - 2(\qquad\qquad)$
30. $a^2 + ab - ac - bc = a(\qquad\qquad) - c(\qquad\qquad)$

Chapter 11

SIMPLE EQUATIONS

In Chapter 3 we saw one way in which to solve easy equations. Since the unknown number we were trying to find was always of the first degree in the equation, the equation was called a *simple equation*. We now proceed to show the general way in which such an equation can be solved.

In an equation such as $3x - 2 = 8 - x$, we speak of the two parts, on either side of the sign $=$, as the sides of the equation.

The left side is the number $3x - 2$.

The right side is the number $8 - x$.

The equation states that these two numbers are equal.

When two numbers are equal, if we:

(1) *add the same number to each,*
(2) *subtract the same number from each,*
(3) *multiply each by the same number.*
(4) *divide each by the same number,*

then, in each case, the resulting numbers will be equal.

Thus:

(1) If $x = 8$, adding 2 to each side will leave the resulting numbers equal.

$$\therefore\ x + 2 = 8 + 2$$

(2) Subtracting 2 from each side will also leave the resulting numbers equal.

$$\therefore\ x - 2 = 8 - 2$$

(3) On multiplying both sides by 3, the resulting numbers will be equal.

$$\therefore \quad 3x = 8 \times 3$$
$$\text{i.e.} \quad 3x = 24$$

(4) On dividing each side by 2, the resulting numbers will be equal.

$$\therefore \quad \frac{x}{2} = \frac{8}{2}$$
$$\text{i.e.} \quad \frac{x}{2} = 4$$

Example 1: Solve $x - 5 = 11$.

$$x - 5 = 11$$

Add 5 to each side.

[*Note*. Any number could have been chosen. 5 is chosen so that we will be left with x on the left-hand side.]

$$\therefore \quad x - 5 + 5 = 11 + 5$$
$$\text{i.e.} \quad x = 11 + 5$$
$$\therefore \quad x = 16$$

Check (or Verification)

When $x = 16$,

Left side of equation, i.e. $x - 5 = 16 - 5$
$= 11$

Right side of equation $= 11$

$\therefore$ Left side = Right side

$\therefore$ $x = 16$ is the correct solution.

Example 2: Solve $3x + 4 = 14 - 2x$.

$$3x + 4 = 14 - 2x$$

Add $2x$ to each side.

$$\therefore \quad 3x + 4 + 2x = 14 - 2x + 2x$$
$$\text{i.e.} \quad 5x + 4 = 14$$

Subtract 4 from each side.

$$\therefore\ 5x + 4 - 4 = 14 - 4$$
$$\text{i.e. } 5x = 10$$

Divide both sides by 5.

$$\therefore\ x = 2$$

[*Note* The aim is to collect all terms containing x on the Left side and known numbers (called *constants*) on the Right side.

(1) $2x$ is added to each side to remove $-2x$ from the Right side.
(2) 4 is subtracted from each side to remove 4 from the Left side]

Check

When $x = 2$,

$$\text{Left side} = 3 \times 2 + 4$$
$$= 10$$
$$\text{Right side} = 14 - 2 \times 2$$
$$= 10$$
$$\therefore\ \text{Left side} = \text{Right side}$$
$$\therefore\ x = 2 \text{ is the correct solution.}$$

Exercises 32

Solve, showing all the steps, and check your solutions:

1. $x + 3 = 7$
2. $x - 5 = 9$
3. $3\frac{1}{2} + x = 8$
4. $x - 3 = -4$
5. $x - 1\frac{1}{4} = 6\frac{1}{2}$
6. $4x = 16$
7. $3x = 0$
8. $8x = 24$
9. $3x = -15$
10. $-2x = -12$
11. $20 = -4x$
12. $\frac{x}{5} = 3$
13. $\frac{1}{4}x = -2$
14. $\frac{1}{3}x = -2\frac{1}{6}$
15. $3x + 2 = 15$
16. $5x - 3 = 22$
17. $13 = 3x + 1$
18. $8 - 2x = 3x + 23$
19. $\frac{1}{4}x - 7 = 3$
20. $4 - 2x = x - 2$
21. $0{\cdot}2x = 3$
22. $-2{\cdot}5x = 6{\cdot}25$
23. $2x - 1{\cdot}5 = 5x$
24. $6x - 17 = 4 - 3x$
25. $6x - 15 - 2x + 25 = 0$

Consider the equation

$$2x - 5 = 9 + x \qquad (1)$$

In solving it we first subtract x from each side.

$$\therefore\ 2x - 5 - x = 9 + x - x$$
$$\text{i.e. } 2x - 5 - x = 9 \quad \dots \quad (2)$$

If we compare equation (2) with equation (1), we see that equation (1) has two terms on each side, whereas equation (2) has three terms on the left side and one term on the right side, and also that equation (2) could have been obtained from equation (1) by moving the term $+x$ from the right side to the left side, and at the same time changing its sign.

If to each side of equation (2) we add 5, we get

$$2x - 5 - x + 5 = 9 + 5$$
$$\text{i.e. } 2x - x = 9 + 5 \quad \dots \quad (3)$$

Equation (3) could have been obtained from equation (2) by moving the term -5 from the left side to the right side, and at the same time changing its sign.

Hence any term of an equation can be moved from one side of the equation to the other, provided the sign of the term is changed. This moving of a term from one side to the other is called *transposing* the term.

Note: If in the course of solving any equation we obtain an equation such as $16 = 4x$ we can write down *at once* $4x = 16$.

Example: Solve $8x - 10 = 5 + 2x$.

$$8x - 10 = 5 + 2x$$

Transpose the x term to the left side, and the constant term to the right side.

$$\therefore\ 8x - 2x = 5 + 10$$
$$6x = 15$$
$$\therefore\ x = \frac{15}{6}$$
$$= 2\tfrac{1}{2}$$

Check.

When $x = 2\frac{1}{2}$,

$$\begin{aligned} \text{Left side} &= 8 \times 2\tfrac{1}{2} - 10 \\ &= 20 - 10 \\ &= 10 \\ \text{Right side} &= 5 + 2 \times 2\tfrac{1}{2} \\ &= 5 + 5 \\ &= 10 \end{aligned}$$

$\therefore$ Left side = Right side

$\therefore$ $x = 2\frac{1}{2}$ is the correct solution.

Exercises 33

Solve, and check the results in each case:

1. $3x + 4 = 10$
2. $2x - 5 = 7$
3. $3 + 7x = 10$
4. $5y + 10 = 0$
5. $4t - 6 = 0$
6. $2n + 5 = n + 7$
7. $5n - 3 = 4n + 8$
8. $7 - 3x = 2x + 12$
9. $5 + 7x = 8 - 5x$
10. $3x + 4 = 4 - 9x$
11. $3x - 9 = 21 - 7x$
12. $18 + 5x = 8x + 33$
13. $2x - 5 = 6x - 14$
14. $5x + 12 = 3x - 5$
15. $4 + 3y = 2 - 5y$
16. $15 + 6x - 19 = 3x$
17. $2(x + 3) = 10$
18. $5(1 - 2x) = 15$
19. $3(t - 2) = 0$
20. $4(5 + 3x) = 2x$
21. $1 + 2(x + 3) = 7$
22. $3(7x - 4) + 2x = 11$
23. $5 + 4x = -11(5 + x)$
24. $4(2 - x) = 5(1 - 2x)$
25. $7x - 6(x - 5) - 42 = 0$
26. $3x + (2x - 5) - (3x + 2) = 0$
27. $2(x - 1) - 3(2x - 5) = 3$
28. $3t + 5(8 - t) = 24 - t$
29. $10(2x + 3) - 8(3x - 5) + 5(2x - 8) = 0$
30. $6(x + 2) - 3(x - 2) = 18$
31. $2(y + 3) = 3(2y - 5) + 5(y - \frac{3}{5})$
32. $9x + 4(1 - 3x) = -4 - 6(x - 1)$
33. $3(x - 2) - 4(x - 1) = 2(4 - x) - 7$
34. $5(4 - 3x) - 2(10 - x) = 5x$

35. $3(2x - 3) - 2(5 - 4x) = 2x - 1$
36. $5x - 4(x + 2) = 8 - (2 - 5x)$
37. $6(x - 3) - 4(x - 2) = 37 - 3(x + 4)$
38. $2(x - 1\frac{1}{2}) = 6x + 30(x - 12)$
39. $3(2x - 1) - 4(x + 3) = 2(4 - x)$
40. $x(2x + 5) - 2x(x - 3) = 22$

Prove that:

41. $4x + 8 = 3x + 10$ when $x = 2$
42. $19 + 6x = 2x + 7$ when $x = -3$
43. $6x - 13 = 12 - 5x - 8\frac{1}{2}$ when $x = 1\frac{1}{2}$
44. $4(x - 3) = 6(2x - 1)$ when $x = -\frac{3}{4}$
45. $24 - 4(x - 2) = 18 - 2x$ when $x = 7$
46. $3(20 + 3x) - 5(2x + 13) = 0$ when $x = -5$
47. $6 - 3(4 - y) = 10 - 2(3y - 1)$ when $y = 2$
48. $2(u - 3) = 2u + 27 - 6(2 - u)$ when $u = -3\frac{1}{2}$
49. $t - 1 = 2(3 - t) - 6 - 2t$ when $t = \frac{1}{5}$
50. $6(3x - 5) + 5(6 - 5x) = -9x - 12$ when $x = -6$

Equations with Fractions

Example: Solve $2x + 7 = \dfrac{4x + 5}{3} - \dfrac{x - 5}{5}$.

We begin by clearing of fractions, and to do this multiply both sides by 15, the L.C.M. of the two denominators 3 and 5.

$$\therefore\ 2x \times 15 + 7 \times 15 = \frac{4x + 5}{\cancel{3}_1} \times \frac{\cancel{15}^5}{1} - \frac{x - 5}{\cancel{5}_1} \times \frac{\cancel{15}^3}{1}$$

$$\therefore\ 30x + 105 = 5(4x + 5) - 3(x - 5)$$

$$\therefore\ 30x + 105 = 20x + 25 - 3x + 15$$

Transposing,

$$\therefore\ 30x - 20x + 3x = 25 + 15 - 105$$

$$13x = -65$$

$$x = -5$$

Check.

When $x = -5$,

$$\begin{aligned}\text{Left side} &= 2(-5) + 7\\ &= -10 + 7\\ &= -3\\ \text{Right side} &= \frac{4(-5) + 5}{3} - \frac{(-5) - 5}{5}\\ &= \frac{-20 + 5}{3} - \frac{-10}{5}\\ &= \frac{-15}{3} - (-2)\\ &= -5 + 2\\ &= -3\end{aligned}$$

$\therefore$ Left side = Right side.

$\therefore$ $x = -5$ is the correct solution.

Exercises 34

Solve:

1. $\frac{x}{2} + \frac{x}{3} = 5$
2. $\frac{x}{3} - \frac{x}{4} = 2$
3. $\frac{x}{4} + \frac{x}{2} = 1\frac{1}{2}$
4. $\frac{1}{5}x - \frac{1}{6}x = -\frac{1}{3}$
5. $\frac{x}{2} - \frac{x}{3} - \frac{x}{4} = 1$
6. $\frac{2}{3}x - \frac{3}{8}x = 7$
7. $\frac{7}{2}x - \frac{3}{4}x - 2x = 18$
8. $\frac{y}{6} - 5 = \frac{y}{5} - 7$
9. $\frac{2}{3}x + 41 = 18 - \frac{3}{7}x$
10. $\frac{x}{3} - \frac{3x}{5} = 6\frac{3}{5} - x$
11. $\frac{1}{4} + \frac{11}{x} = \frac{4}{5}$
12. $\frac{3}{x} + \frac{2}{x} = 5$
13. $\frac{5}{x} - \frac{2}{x} = 6$
14. $\frac{4}{3x} - 2 = 5 - \frac{8}{x}$
15. $\frac{6x + 3}{7} - \frac{5x - 6}{3} = 0$
16. $\frac{3}{y} - \frac{9}{16} = \frac{5}{2y} - 1$
17. $\frac{3 + y}{4} = \frac{1 - 3y}{3}$
18. $\frac{x}{2} + \frac{x - 2}{3} = 4$

19. $\dfrac{5}{4x} - \dfrac{1}{5x} = 3$

20. $\dfrac{x}{4} - \dfrac{2x - 3}{6} = 2$

21. $\dfrac{x + 3}{4} - \dfrac{x}{3} = \dfrac{11}{15}$

22. $\frac{1}{3}(4x - 9) - \frac{1}{6}(5x + 12) = 3x$

23. $\frac{1}{2}(x - 4) - \frac{1}{3}(2x - 3) + \frac{1}{4}(x + 4) = 0$

24. $\dfrac{x + 2}{3} - \dfrac{x - 2}{2} = 1 - \dfrac{2x}{3}$

25. $\dfrac{2x - 1}{3} - \dfrac{x + 3}{2} = x - 2$

26. $\dfrac{3x - 2}{2} = \dfrac{1 - 5x}{3} + 5$

27. $\dfrac{x - 5}{4} - \dfrac{3 - 4x}{9} = 13$

28. $\dfrac{3 - 7x}{10} + \dfrac{3x - 2}{5} = x - \dfrac{1}{10}$

29. $\dfrac{y - 1}{4} - \dfrac{5y + 2}{5} = \dfrac{7}{20}$

30. $\dfrac{4x - 1}{4} - \dfrac{2x - 3}{10} = \dfrac{9x - 6}{5}$

31. $\dfrac{2x - 1}{6} - \dfrac{3x + 4}{15} = \dfrac{x - 4}{30}$

32. $\dfrac{x}{4} - \dfrac{3 - 2x}{3} - \dfrac{x}{2} + 11 = 0$

33. $\dfrac{3x - 7}{10} + \dfrac{x - 3}{3} = \dfrac{4x}{9}$

34. $\dfrac{3t - 1}{5} - \dfrac{2t + 3}{3} = t$

35. $3x + 13 - \dfrac{2x + 1}{3} + \dfrac{x}{5} = 0$

36. $\dfrac{2x}{3} - 11 = 20 - \dfrac{x}{5} - \dfrac{x - 10}{4}$

37. $\dfrac{5x - 1}{6} - \dfrac{5 + x}{9} = \dfrac{2x - 3}{3}$

38. $\dfrac{4(x-1)}{5} - 4 = \dfrac{7(x-4)}{10} - \dfrac{3}{5}$

39. $\dfrac{2u-4}{7} - \dfrac{2u-26}{3} = 5 - \dfrac{3u+5}{2}$

40. $\frac{1}{2}(2x + \frac{1}{3}) - \frac{1}{3}(\frac{1}{2} - 2x) = \frac{1}{4}(2x + 1)$

41. $\dfrac{5x-2}{3} + x = \dfrac{1-4x}{2} + 1\frac{1}{6}$

42. $\frac{2}{3}(3x - 1) + \frac{1}{4}(2 - 5x) - \frac{3}{5}(x + 5) + 4 = 0.$

43. $0{\cdot}2t - 3{\cdot}5 = 2{\cdot}8 - 0{\cdot}5t$

44. $1{\cdot}2(3{\cdot}5 - 2x) = 0{\cdot}6(3x - 7)$

45. $0{\cdot}6(2x + 3) + 2(0{\cdot}3x - 0{\cdot}8) = 7{\cdot}4$

46. $\dfrac{x-2}{0{\cdot}4} - \dfrac{4x-6}{0{\cdot}5} = 29$

47. $0{\cdot}5(2y - 0{\cdot}3) = 0{\cdot}3(3y + 0{\cdot}7) + 0{\cdot}54$

48. $\frac{1}{2}(x - \frac{1}{2}) - \frac{1}{5}(3x - 2) = \frac{3}{4}(x - 1\frac{1}{2})$

Literal Equations

Example: Solve $\dfrac{a(x-b)}{c} = 2.$

In such an equation we regard x as the unknown, and we try to find its value expressed in terms of a, b, c. An equation like this is called a *literal equation*. In such equations the unknowns are usually letters from the end of the alphabet, while the other letters are from the beginning of the alphabet.

$$\frac{a(x-b)}{c} = 2$$

Multiply both sides by c,

$$\therefore \quad \frac{a(x-b)}{\cancel{c}_{1}} \times \frac{\cancel{c}^{1}}{1} = 2 \times c$$

$$\therefore \quad a(x-b) = 2c$$

$$\therefore \quad ax - ab = 2c$$

Transposing,

$$\therefore \; ax = 2c + ab$$

$$\therefore \; x = \frac{2c + ab}{a}, \text{ or } \frac{2c}{a} + b.$$

Check.

When $x = \frac{2c}{a} + b$,

$$\text{Left side} = \frac{a\left(\frac{2c}{a} + b - b\right)}{c}$$

$$= \frac{a\left(\frac{2c}{a}\right)}{c}$$

$$= \frac{2c}{c}$$

$$= 2$$

$$\text{Right side} = 2$$

$\therefore \; x = \frac{2c}{a} + b$ is the correct solution.

Exercises 35

Solve:

1. $x + 5 = a$
2. $x - 3 = b$
3. $bx = a$
4. $ax + b = 0$
5. $\frac{x}{a} = b$
6. $\frac{x}{3} - b = c$
7. $5x + a = b$
8. $3x - c = ab$
9. $\frac{x - b}{c} = a$
10. $\frac{2x + b}{a} = c$
11. $a(x - b) = bc$
12. $\frac{x + b}{a} = \frac{c}{a} + 1$
13. $\frac{a}{x} + \frac{b}{x} = 1$
14. $\frac{a(x + b)}{c} = d$

REVISION PAPERS 11–15

Paper 11

1. Simplify, and arrange the result in ascending order, $3x^2(2 - x^2) - 4x(x + 3) + (2x^3 - 5)$.

What is the value of the answer when $x = -1$?

2. Simplify:

(i) $\dfrac{ab}{c} \times \dfrac{a^2}{bc}$ (iii) $\dfrac{6a^3}{(-b)^3} \times \dfrac{-b}{2a^2} \div \dfrac{9a}{b}$

(ii) $2(ab^2)^3 \div (a^2b)^2$

3. Solve:

(i) $3x + 7 = 2 - x$

(ii) $5(t - 3) - 3(5 - 3t) - 8(4t - 5)$

4. Simplify:

(i) $\dfrac{a}{2} + \dfrac{a}{3} - \dfrac{a}{4}$ (iii) $1 - \dfrac{3a + 2}{15} + \dfrac{a}{3}$

(ii) $\dfrac{2a}{3} - \dfrac{3a}{4} + \dfrac{4a}{5}$

5. Simplify:

(i) $5[3x - 2(x - y)]$

(ii) $5[3(x - \overline{2x - y})]$

6. A man sets out on a journey of 100 miles. He travels for 3 hr. at a m.p.h. What must his speed be in m.p.h. if he is to complete his journey in 2 hr.? If his speed is $30\frac{1}{2}$ m.p.h., find the value of a.

Paper 12

1. Simplify $6x - 3[2x - 4\{5 - (x - 3)\} + 15]$.
Find the value of the answer when $x = -2$.

2. Simplify:

(i) $x - \dfrac{3x}{4} + \dfrac{x}{5}$ (iii) $\dfrac{3}{a} + \dfrac{2}{3b} - \dfrac{5}{a^2}$

(ii) $\dfrac{3x}{4y} - \dfrac{x}{2y} + \dfrac{1}{y}$

3. Solve:

(i) $\dfrac{3x}{4} = 1 - \dfrac{2 - 3x}{5}$

(ii) $\frac{2}{3}(4x + 5) - \frac{1}{2}(2 - 5x) + 8 = 0$

4. Simplify:

(i) $\dfrac{a^3b \times -bc^2}{-a^2b^2c^2}$

(ii) $\dfrac{(-3x)^2}{y^2} \times \dfrac{-y}{3x} \times \dfrac{y}{x}$

(iii) $\left[\left(-\dfrac{x}{y}\right)^2 \div -\dfrac{x}{y}\right] + \left[\left(\dfrac{x}{y}\right)^2 \div \dfrac{x}{y}\right]$

5. If $y = 5 + 2x - 3x^2$, complete the following table giving values of y for various values of x:

x	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	4
y						

6. A man buys x jotters for a shillings, and sells them at b pence each. Find an expression for his gain in pence.
What fraction is this gain of his Cost Price?
Find the value of this fraction if $x = 60$, $a = 30$, $b = 8$.

Paper 13

1. If $x = 3a$, $y = -2a$, find, in terms of a, the value of:

   (i) $(2x + y)(x - 3y)$

   (ii) $\sqrt{(x - y)(3x + 2y)}$

   (iii) $\dfrac{6x + 3y}{7x - 3y}$

2. Solve:

   (i) $x = 0{\cdot}3x - 1{\cdot}4$

   (ii) $\frac{1}{3}(3x - 2) - \frac{1}{5}(x - 3) = 1$

3. Simplify:

   (i) $\dfrac{1}{3n} - 2 + \dfrac{3}{n}$

   (ii) $\dfrac{5}{2a} + \dfrac{3}{4a} - \dfrac{8}{5a}$

   (iii) $\dfrac{4}{a^2} - \dfrac{3}{a} + 2$

4. From the sum of $5a^2 - 4ab + 2b^2$ and $3ab - 2a^2 - 3b^2$ take the sum of $5b^2 - 2a^2$ and $6ab - 3b^2$.

5. If $x = -2$, $y = -3$, $z = 1$, find the value of:

   (i) $(3y - z)^2 \div (2x)^3$

   (ii) $y^3 - 3xyz$

   (iii) $x^2 + y^2 - xy - yz$

6. If tea costs x shillings per lb., and coffee is 2 shillings per lb. dearer than tea, find an expression for the cost of 5 lb. of tea and 6 lb. of coffee. If the total cost is £5, find the price of 1 lb. of tea.

Paper 14

1. When $a = -3$, $b = 2$, $c = -1$, find the value of:

   (i) $a^2 + b^2 + c^2 - ab - bc - ca$

   (ii) $\dfrac{a^3 + b^3 + c^3 - 3abc}{a + b + c}$

2. Simplify:

   (i) $\left(\dfrac{2a^2}{b} - \dfrac{3b}{c} + bc\right) \times (-bc)$

   (ii) $\dfrac{2a^2}{3b^2} \times \dfrac{b}{a} - \dfrac{3a}{2b}$

   (iii) $\dfrac{x}{4x} + \dfrac{y}{2y^2} - \dfrac{3z}{9yz}$

3. Two sides of an equilateral triangle are respectively $(3x - 11)$ in., and $(22 - 3x)$ in.

Find the perimeter of the triangle, in inches.

4. Solve:

(i) $5{\cdot}6x - 1{\cdot}9 = 3{\cdot}2x + 3{\cdot}7$

(ii) $0{\cdot}5(x - 0{\cdot}2) - 1{\cdot}5(2x - 0{\cdot}4) = 2{\cdot}5(3x + 1)$

5. If $x^2 - 3x + a = -2$ when $x = 1$, find a, and then find the value of the expression on the left-hand side, when $x = -\frac{1}{2}$.

6. Milk is bought at a shillings per gal., and sold at b pence per pint. Find an expression for the gain, in shillings, on c quarts.

Paper 15

1. A rectangle is $(2x + 3)$ in. long, and $(3x - 5)$ in. broad. If its perimeter is 2 ft. 2 in., find the value of x.

What are the dimensions of the box?

2. Solve $\frac{2x + 3}{5} - \frac{3x - 4}{4} = 1 - \frac{x}{3}$.

3. Simplify:

(i) $a(b + c - a) - b(c + a - b) - c(a - b - c)$

(ii) $3x(x^2 - 2x + 1) - x^2(x + 5) - 3(x - 2)$

4. (i) A wire l in. long has a part cut off and a length a in. remains. What fraction of the whole wire was cut off?

(ii) To x gal. of milk, y gal. of water are added. What fraction of the mixture is milk? How much milk would there be in 2 gal. of the mixture?

5. If $3(2x - y) = 4 - y(x + 9)$ when $y = 2$, find the value of $1 + 2x - 3x^2$.

6. A man buys x articles at s shillings each. He sells a quarter of them at double their cost price, half of them at one and a half times their cost price and the rest at half their cost price. Find an expression for the total gain in shillings, and simplify it.

CHAPTER 12

SYMBOLICAL EXPRESSION

Example 1: If x is an even number, write down the next two even numbers greater than x, and the next two even numbers less than x.

If may help you to solve a problem such as this, if you think of any even number in place of x.

Suppose, for example, you think of 8. The next two even numbers greater than 8 are 10 and 12, and you know that 10 is $8 + 2$, and that 12 is $8 + 4$, so that the next two even numbers greater than 8 can be got by adding 2 and 4 to 8. If you now replace 8 by x, the next two even numbers greater than x will be got by adding 2 and 4 to x. Therefore they are $(x + 2)$ and $(x + 4)$.

In exactly the same way, the next two even numbers less than 8 are found to be $8 - 2$ and $8 - 4$. Therefore the next two even numbers less than x are $(x - 2)$ and $(x - 4)$. The five even numbers beginning with the least would be $(x - 4)$, $(x - 2)$, x, $(x + 2)$, $(x + 4)$.

Note: This method of working with a known number in place of a letter is often helpful at the beginning. Even so, only with experience will you get to know a suitable number to choose. For example, in the problem above it would probably not have been helpful in the second part of the problem if you had chosen 2 as the even number

Later, of course, you will be able to work such problems using letters only.

Example 2: If y exceeds a certain number by x, what is the number?

Think of the number 12 say for y, and 5 for x. The question now reads: If 12 exceeds a certain number by 5, what is the num-

ber? Now 12 exceeds 7 by 5, so that the number is 7, and 7 is got from the two numbers 12 and 5 by subtracting 5 from 12.

Replacing 12 by y and 5 by x, the answer to the original question is therefore $(y - x)$.

Example 3: A man buys a case of apples containing x stones for y shillings. He sells them at b pence per lb. Find his gain in shillings.

To find the gain we must find the total cost price and the total selling price.

$$\begin{aligned}
\text{Total Cost Price} &= y \text{ shillings}\\
\text{Selling Price of 1 lb.} &= b \text{ pence}\\
\text{,,} \quad 1 \text{ stone} &= 14b \text{ pence}\\
\text{,,} \quad x \text{ stones} &= 14bx \text{ pence}\\
&= \frac{14bx}{12} \text{ shillings}\\
\text{Gain} = \text{Selling Price} - \text{Cost Price} &\\
&= \frac{14bx}{12} \text{ shillings} - y \text{ shillings}\\
&= \left(\frac{14bx}{12} - y\right) \text{ shillings}\\
&= \left(\frac{7bx}{6} - y\right) \text{ shillings}
\end{aligned}$$

Exercises 36

1. Write down three consecutive whole numbers (i.e. integers), if the least one is x.

2. Write down three consecutive integers, if the greatest one is x.

3. Write down three consecutive odd numbers, the least being x. Find their sum.

4. Write down three consecutive odd numbers, the greatest being x.

5. Write down three consecutive integers, the middle one being n. Find their sum.

6. If $n - 4$ is an even number, what is the next higher even number?

7. Write down three consecutive even numbers, if the middle one is $n - 1$.

8. If n is any integer, what kind of number must $2n$ be?

9. If n is any integer, what kind of number must $(2n + 1)$ be?

10. If the sum of two numbers is x, and one of them is 7, what is the other?

11. If the difference of two numbers is y, and the smaller one is 3, what is the other?

12. If the sum of two numbers is x, and one of them is y, what is the other?

13. x is greater than a certain number by 8. What is the number?

14. What number exceeds x by y?

15. The product of two numbers is 15. One of them is x. What is the other?

16. When 20 is divided by a certain number, the answer is q. What is the number?

17. When a certain number is divided by 5, the quotient is a. What is the number?

18. (i) What number, divided by 3, gives a quotient 5 and a remainder 2?

(ii) What number, divided by 3, gives a quotient q and a remainder 2?

(iii) What number, divided by a, gives a quotient q, and a remainder r?

19. If a number n is multiplied by 3, and 5 is added, what is the result?

20. Think of a number n. Add 3. Double your answer, and divide the result by 5. What is the final answer?

21. Think of a number n. Double it. Add 3, and divide the answer by 5. What is the final answer?

22. Subtract 3 from x, multiply the answer by 5, add 8 to the result, and divide the answer by 2.

23. What fraction is obtained by subtracting 5 from the numerator, and from the denominator, of the fraction $\frac{m}{n}$?

24. The number $34 = 3 \times 10 + 4$.

What is the number which has 3 in the tens place, and x in the units place? [Note that this cannot be written $3x$.]

25. What is the number, if x is in the tens place, and y in the units place?

26. What is the number which has x in the hundreds place, y in the tens place, z in the units place?

27. If a number c is in the units place, and b in the tens place, what is the number?

What number would be obtained by reversing the digits?

28. In a school with x pupils on the roll, there are b boys. How many girls are there? How many more boys are there than girls?

29. A man does a journey of y miles by walking x miles, travelling z miles by train and taking a bus for the remainder of the journey. How far does he travel by bus?

30. The average of a set of numbers is k. If there are n numbers in the set, and the number 10 is withdrawn, find the new average of the remaining numbers.

---

31. If two boys have x shillings between them, and one has 5*s*., how much has the other?

32. £1 is divided among three boys. One gets x shillings, the second $3x$ shillings. How many shillings has the third?

33. Coffee costs x pence per lb., and tea y pence per lb. Find, in shillings, the cost of a lb. of tea and b lb. of coffee.

34. Coffee costs x shillings per lb., and tea half as much as coffee. Find, in shillings, the total cost of a half-lb. packets of coffee, and b quarter-lb. packets of tea.

35. A man buys a 1-cwt. bags of potatoes, at s shillings per bag. He sells all the potatoes at $\frac{1}{a}$ shillings per lb. Find his gain in shillings.

36. A motorist reckons that his car does k miles to the gallon of petrol, the petrol costing 5*s*. per gal. He estimates that other charges amount to y pence per mile. Find, in £, the total cost for a journey of 480 miles.

37. A man buys x apples at a for a shilling, and sells them at b pence each. Find his gain, in shillings.

38. A boy has enough money to buy 20 bars of chocolate at x pence per bar. How many more bars can he buy, if the price of each bar is reduced by 1*d*?

39. The total number of tickets sold for a concert is $6x$. Of these x cost 5*s*. each, $2x$ cost 3*s*. 6*d*. each, and the rest 2*s*. each. Find the total drawings, in £.

40. A man mixes two kinds of tea, x lb. at 5*s*. per lb. with $5x$ lb. at 4*s*. per lb. He sells all the tea at 6*s*. per lb. Find his gain, in £.

41. A single ticket to a certain place costs x shillings, and a return ticket $1\frac{3}{4}$ times as much as a single ticket. Find, in shillings, the cost of:

(*a*) 6 single tickets and 4 return tickets.
(*b*) 4 ,, ,, 6 ,,

42. Divide £100 between 2 men, A and B, so that A gets £x less than B.

43. Divide £x between two boys, so that, for every 2*s* 6*d*. one gets, the other gets 2*s*.

44. A man bought a articles at £x each. He sold $\frac{3}{4}$ of them at £$1\frac{1}{2}x$ each, and the remainder at half their cost price. Find his total profit, in £.

45. Divide £$3x$ among A, B and C so that A gets £10 more than B, and B £10 more than C.

46. A collection, amounting to £10, consisted of florins, shillings and sixpences. There were x florins, and half that number of shillings. How many sixpences were there?

47. A man buys an article for £x, and sells it for £5. What fraction of the cost price does he gain? Express this as a percentage.

48. In his will a man left half his estate to his wife, $\frac{1}{5}$ to each of his two sons and the rest to charity. If £x was left to charity, what was the value of the estate?

49. For the loan of books a library charges $4d.$ per volume for the first week, and $2d.$ per week thereafter. A man borrows 3 books, and, when he returns them, has to pay x shillings. For how many weeks did he keep the books? Check your answer by putting $x = 5$.

---

50. Two boxes weigh x lb. One is 5 lb. heavier than the other. Find the weight of each.

51. A truck, loaded with 1-cwt. bags of coal, weighs x tons altogether. If there are y bags of coal in the truck, what is the weight of the truck, in tons?

52. A crate contains $\frac{1}{4}$-lb. tins of cocoa. The weight of the crate and its contents is y lb., the crate alone weighing x lb. How many tins of cocoa are in the crate?

53. A bottle contains x oz. of water. If 1 pint of water weighs $1\frac{1}{4}$ lb., what is the capacity of the bottle in pints?

54. g gal. of milk are sent to a school, where there are x pupils. Some of the pupils get $\frac{1}{3}$ pint milk each, and all the milk is used. How many pupils do not get milk?

55. A clock gains x min. every 24 hr. If it is y min. slow at present, in how many hours will it show the correct time?

56. A rectangle is twice as long as it is broad. If it is x ft. broad, find its perimeter, in yd.

57. What length of wire (in miles) will be required to put a four-strand wire fence round a square field, the length of each side of the field being $10x$ chains?

58. The scale of a map is 1 : 10560. What is the actual distance, in miles, between two places that are x in. apart on the map?

59. From a piece of cloth y yd. long, 24 pieces, each y in. long, are cut. What length (in ft.) remains?

60. A metal bar l ft. long increases to a length of l ft. x in. when its temperature is raised 100 Centigrade degrees. By what fraction of its original length does it expand, for 1 Centigrade degree rise in temperature?

61. If 25 litres = 44 pints, how many gallons are equal to $20x$ litres?

62. The perimeter of a rectangle is $2x$ yd. Its length is $2x$ ft. Find its breadth in inches.

63. Two rectangles have the same perimeter. The first is $5x$ in. long and $2x$ in. broad. The second is $4x$ in. long. Find the difference in their areas.

64. x c.c. of a liquid which weighs 1·4 gm. per c.c. is mixed with $2x$ c.c. of another which weighs 0·8 gm. per c.c. What is the weight of 1 c.c. of the mixture?

65. A man is x years old. How old was he 5 years ago? How old will he be in 5 years?

66. A boy is x years old. How many years ago was he 5 years old?

67. A boy is 3 years older than his sister. If he is $(x - 5)$ years of age, in how many years will she be x years old?

68. A man is 24 years older than his son, who is x years old. Find the sum of their ages.

69. A man is 3 times as old as his son, who is x years old. Find the difference of their ages.

70. In 10 years a man will be 3 times as old as his son is now. If his son is x years now, find the father's present age.

71. A man travels x miles at 30 m.p.h. How long does his journey take?

72. A man takes y hr. to travel 50 miles. Find his speed, in miles per hour.

73. How many minutes will a man take, walking at x m.p.h., to walk $\frac{1}{2}$ mile?

74. How many miles will an aeroplane travel in 36 sec. if its speed is x m.p.h.?

75. A man walks for x min. at 4 m.p.h. and runs for the same time at double that speed. Find the total distance he covers (in yards).

76. An aeroplane travels 800 miles due East. In still air its speed is x m.p.h. and the wind is blowing from West to East at 30 m.p.h. How long will the journey take? How long will the return journey take if the wind remains constant in speed and direction?

77. How long (in seconds) will it take a train, travelling at 45 m.p.h., to pass a signal, if the train is x yd. long?

78. A man expects to do a journey of x miles, at an average speed of 30 m.p.h., to just keep an appointment. His average speed is actually 2 m.p.h. less than he expects. How many minutes late is he?

Chapter 13

PROBLEMS INVOLVING SIMPLE EQUATIONS

Example 1: A club's annual subscription was £10 for town members and £6 for country members. If there were 90 more town members than country members, and the yearly income from subscriptions was £2100, how many town members were there?

Let x = number of town members

Then $(x - 90)$ = „ country members

Total yearly subscriptions of town members

$$= £10 \times x = £10x$$

Total yearly subscriptions of country members

$$= £6 \times (x - 90) = £6(x - 90)$$

$$\therefore \text{ Total yearly subscriptions} = £10x + £6(x - 90)$$

$$\text{But this total} = £2100$$

$$\therefore £10x + £6(x - 90) = £2100$$

$$\therefore 10x + 6(x - 90) = 2100$$

$$\therefore 10x + 6x - 540 = 2100$$

$$\therefore 10x + 6x = 2100 + 540$$

$$\therefore 16x = 2640$$

$$\therefore x = 165$$

Number of town members = 165.

Example 2: A collection consists of half-crowns and florins and amounts to £8 2*s*. 6*d*. The number of florins was double the number of half-crowns. How many coins were there of each kind?

Let x = number of half-crowns

$\therefore 2x$ = „ florins

$\therefore$ Total collection = x half-crowns + $2x$ florins

But ,, $= £8\ 2s.\ 6d.$

$\therefore$ x half-crowns $+ 2x$ florins $= £8\ 2s.\ 6d.$

$\therefore$ $5x$ sixpences $+ 8x$ sixpences $= 325$ sixpences

$$\therefore\ 5x + 8x = 325$$

$$\therefore\ 13x = 325$$

$$\therefore\ x = 25$$

$\therefore$ Number of half-crowns $= 25$.

$\therefore$ Number of florins $= 25 \times 2 = 50$.

Example 3: An aeroplane did a certain journey between two owns at an average speed of 500 m.p.h. On the return journey its average speed was decreased by 100 m.p.h. If it took 45 min. onger on the return journey, find the distance between the towns.

Let x miles $=$ distance between the towns

Time taken for journey of x miles at 500 m.p.h. $= \frac{x}{500}$ hr

,, ,, ,, ,, 400 ,, $= \frac{x}{400}$ hr

$\therefore$ $\frac{x}{400}$ hr. is greater than $\frac{x}{500}$ hr. by 45 min.

$$\therefore\ \frac{x}{400}\text{ hr.} - \frac{x}{500}\text{ hr.} = 45\text{ min.}$$

$$\therefore\ \frac{x}{400}\text{ hr.} - \frac{x}{500}\text{ hr.} = \frac{3}{4}\text{ hr.}$$

$$\therefore\ \frac{x}{400} - \frac{x}{500} = \frac{3}{4}$$

Multiply both sides by 2000,

$$\therefore\ 5x - 4x = 1500$$

$$\therefore\ x = 1500$$

$\therefore$ Distance between the towns $= 1500$ miles.

Note: In solving problems:

1. Choose a letter, say x, to stand for a *number* in the problem which is unknown. Do not let the letter stand for a number of things, i.e. a quantity. For instance, in Example 3 above it would be wrong to say: Let $x =$ the

distance between the towns, because the x might refer to miles or furlongs or yards, etc. Similarly, if in a problem the question was "Find the cost of the tea", it would be wrong to say: Let $x =$ the cost of the tea. The x must refer to pounds, shillings or pence, etc., and the weight of tea must be stated. A correct statement would be: Let x shillings = the cost of 1 lb. of tea.

2. As shown above, in Example 2,

$$x \text{ half-crowns} + 2x \text{ florins} = £8\ 2s.\ 6d.$$

is a correct equation, but, before proceeding to solve it, we express all the terms in the same unit:

$$5x \text{ sixpences} + 8x \text{ sixpences} = 325 \text{ sixpences}$$

and then we can write an equation containing numbers only, namely:

$$5x + 8x = 325$$

Exercises 37

1. The sum of two consecutive numbers is 33. Find them.

2. The sum of three consecutive numbers is 54. Find them.

3. A certain number is doubled and 6 is added. The result is 30. Find the number.

4. From five times a certain number 10 is subtracted. The result is 55. Find the number.

5. The sum of three consecutive odd numbers is 63. Find them.

6. Two-thirds of a number exceeds a quarter of it by 15. Find the number.

7. The sum of two numbers is 63. The first is 6 times the second. Find the numbers.

8. The result of subtracting 19 from three times a certain number is the same as subtracting the number from 37. Find the number.

9. The sum of three numbers is 66. The second number is twice the first, and the third exceeds the second by 6. Find the numbers.

10. A certain number is multiplied by 3 and 13 subtracted from the result. The answer is now divided by 7, and the final answer is 5. What is the number?

11. The difference of two numbers is 13. Their sum is 35. What is the smaller number?

12. From the numerator and denominator of the fraction $\frac{17}{21}$ the same number is subtracted and the resulting fraction is $\frac{1}{3}$. What is the number?

13. What number added to the numerator and to the denominator of the fraction $\frac{1}{6}$ makes it equal to $\frac{9}{10}$?

14. Prove that, if you think of any number, multiply it by 4, add 6, divide by 3, subtract 2 and take $\frac{3}{4}$ of the result, the final answer will be the number itself.

15. One boy has 5 times as many marbles as a second boy. If the first were to give 18 marbles to the second, each would then have the same number. How many marbles has each?

---

16. A man bought x lb. tea at 6*s.* per lb., and $(x + 5)$ lb. tea at 5*s.* per lb. The total cost was £9 10*s.* Find x.

17. A man bought 25 plants, some at 1*s.* 6*d.* each, and the rest at 1*s.* each. The total cost was 33*s.* How many did he buy of the dearer kind?

18. A certain number of eggs was bought at 4*s.* per dozen, and three times as many at 3*s.* per dozen. If the total bill was £3 5*s.*, how many dozen eggs were bought altogether?

19. Two boys have 6*s.* 8*d.* between them. If the first gave the second 1*s.* 6*d.* they would each have the same amount. How much has each?

20. Divide a guinea between two boys so that the second has 3*s.* 6*d.* more than the first.

21. A and B together have £35. If A gave B £14 he would then have £5 less than B. How much has each at present?

22. Divide £460 among A, B and C so that B has £50 more than A, and C as much as A and B together.

23. After spending $\frac{2}{3}$ of his money a boy found that he had left 5*s*. more than $\frac{1}{4}$ of his money at first. How much money had he?

24. A man buys 46 books, some at 12*s*. 6*d*. each, some at 10*s*. each. The total bill was £25. How many of each kind did he buy?

25. A man buys x articles for £10. He sells a quarter of them at 8*s*. each, 30 at 4*s*. each, and the rest at 2*s*. each. His total profit is £3 10*s*. Find x.

26. s florins + (s + 11) shillings = £2 10*s*. Find s.

27. A sum of £2 11*s*. consists of half-crowns and shillings. There are 30 coins altogether. How many are there of each kind?

28. A silver collection amounted to £3 12*s*. There were florins, shillings and sixpences, the number of shillings being twice the number of florins, and the number of sixpences twice the number of shillings. How many coins were there of each kind?

29. The subscriptions to a tennis club are: Adults, 5 guineas; Juveniles, 2 guineas. The number of juveniles is 30 less than the number of adults, and the total subscription is £525. How many members are there in the club altogether?

30. At a fair a boy pays 9*d*. for 12 shots at a target. He receives back 6*d*. for each hit. He misses 30 times, but at the end of the shooting finds that he has 9*d*. more than when he started. How many hits did he make?

31. A load of 3 tons is composed of 1-cwt. bags, and 2-stone bags. The number of 2-stone bags is twice the number of 1-cwt. bags. How many bags are there altogether?

32. A man's wage rate is 6*s*. per hour, overtime being counted as time and a half. If in one week his wages were

£15 12*s*., and he worked altogether for 48 hours, how long did he work overtime?

33. Tickets for a concert are priced at 3*s*. 6*d*. and 2*s*. Altogether 260 tickets are bought, and the amount drawn for the dearer seats is 25*s*. less than that drawn for the cheaper ones. What sum was paid for admission?

34. One tank contains 3 times as many gallons of water as another. From the first tank 17 gal. are taken and added to the second tank, and the first now contains 24 gal. more than the second. How many gallons of water did the smaller tank contain at first?

35. Two men A and B have £128 and £200 respectively. If A saves £2 per week and B spends £2 10*s*. per week, after how many weeks will they have the same amount of money?

36. A father is 3 times as old as his son. Their combined ages total 48 years. How old is the son?

37. A father is 23 years older than his son. Their combined ages total 57 years. How old is the father?

38. In 10 years time a father will be twice as old as his son. The son at present is 32 years younger than his father. What is the present age of the father?

39. A father is twice as old as his son. In 10 years time the son will be the same age as the father was 18 years ago. Find the present age of each.

40. At present a man is 3 times as old as his son, aged 12 years. In how many years will he be twice as old as his son, and in how many further years will he be $1\frac{1}{2}$ times as old as his son?

41. A father is 32 years older than his son. 10 years ago he was 3 times as old as his son was then. Find the present age of each.

42. A man cycles for x hr. at 10 m.p.h., and walks for 2 hr. at $(x + 1)$ m.p.h. The total distance he travels is 38 miles. Find x.

43. A boy walks to a village and back in 50 min. He walks there at 3 m.p.h. and returns at 2 m.p.h. How far away is the village?

44. A man walks for 2 hr., cycles for the next 3 hr., and completes his journey by an hour's travel by bus. His rate of cycling is 3 times that of walking, and the bus travels 3 times as fast as he cycles. The total distance covered was 60 miles. Find his rate of walking.

45. A man cycles a certain distance at 9 m.p.h. He increases his speed by $\frac{1}{3}$ on the return journey and finds that he has saved 1 hr. 20 min. Find the distance.

46. One man, by car, takes 3 hr. to do a certain journey, which another man, travelling at 5 m.p.h. less, takes $\frac{1}{2}$ hr. longer to do. Find the length of the journey and the speed of each.

47. A man cycled 65 miles at an average speed of 10 m.p.h. For part of the way he travelled at 12 m.p.h., and for the rest at 9 m.p.h. How long did he travel at the faster speed?

48. Two trains, travelling on parallel lines, meet in 3 hr., the one travelling from A to B, the other from B to A. If the distance from A to B is 300 miles, and one travels 10 m.p.h. faster than the other, find the rate of each.

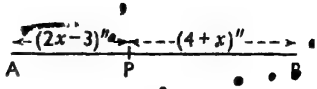
49. A man travels x miles at 10 m.p.h., and then $2x$ miles at 15 m.p.h., resting for $\frac{1}{2}$ hr. between the journeys. If the total time taken was 4 hr., find x.

50. A boy takes a bus from his home to the town library and walks back, taking 18 min. in all. Had he walked both ways he would have taken 12 min. longer. Find the speed of the bus if the boy walks at 4 m.p.h.

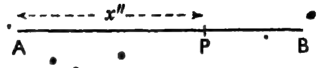
51. On a journey 114 miles a man motors part of the way at 44 m.p.h. and cycles the rest at 16 m.p.h. The time he

spends cycling is double the time he spends motoring. How long did he take for the journey?

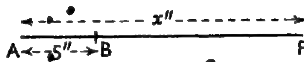
52. AB = 13 in. Find $\frac{AP}{PB}$.



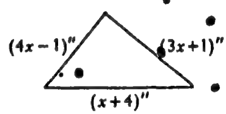
53. AB = 10 in., 2AP = 3PB. Find x.



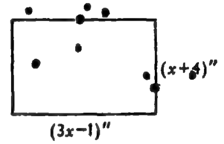
54. If $\frac{AP}{BP} = \frac{4}{3}$, find x.



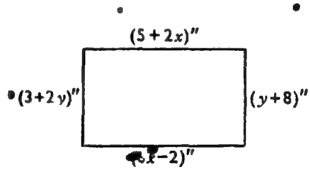
55. If the perimeter of the triangle is 12 in., find x.



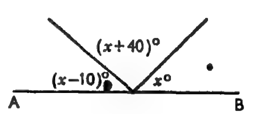
56. If the perimeter of the rectangle is 38 in., find x.



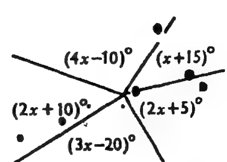
57. ABCD is a rectangle. Find its length and its breadth.



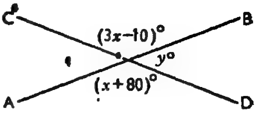
58. AB is a straight line. Find x.



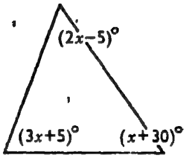
59. Find x.



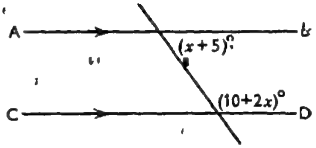
60. Find x and y if AB and CD are straight lines.



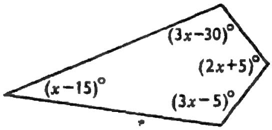
61. Find x.



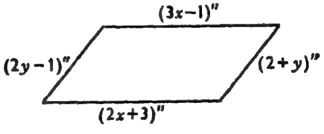
62. AB and CD are parallel lines. Find x.



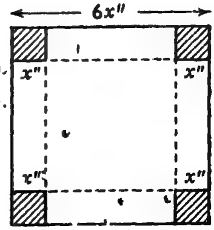
63. Find the angles of the quadrilateral.



64. ABCD is a parallelogram. Find x and y.



65. From the square plate the four equal square corners are cut. If the perimeter of the remaining plate is 48 in., find x.



66. If in the above example a box were made by folding along the dotted lines:

(*a*) What would the dimensions of the bottom of the box be?

(*b*) What would its depth be?

(*c*) What would its volume be?

(*d*) If the volume was 128 cu. in., find the length of the edge of the original square plate.

67. AB and AC are the equal sides of an isosceles triangle. If its perimeter is 18 in., find the length of BC.

68. To 150 c.c. of alcohol (density 0·8 gm. per c.c.) is added sufficient water to make the density of the mixture 0·85 gm. per c.c. If the density of water is 1 gm. per c.c., how much water was added?

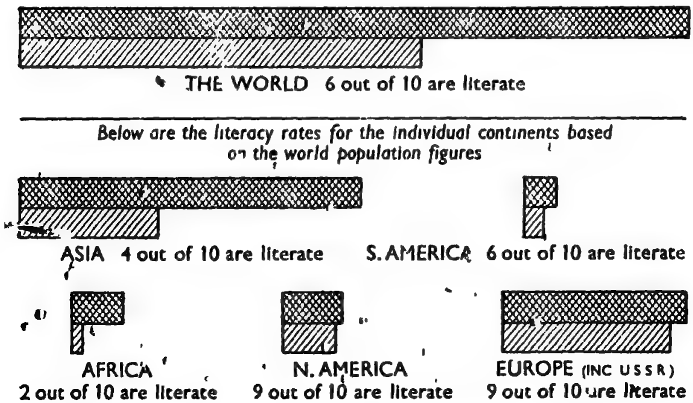
69. With 170 gm. of a metal A (density 8·5 gm. per c.c.) are mixed a certain weight of metal B (density 7 gm. per c.c.). If the density of the resulting alloy is 8 gm. per c.c., find the weight of metal B that was taken.

70. What weight of wood (density 0·75 gm. per c.c.) will it be necessary to fix to a piece of metal weighing 14 gm. (density 2·8 gm. per c.c.) so that the two will just float in water?

CHAPTER 14

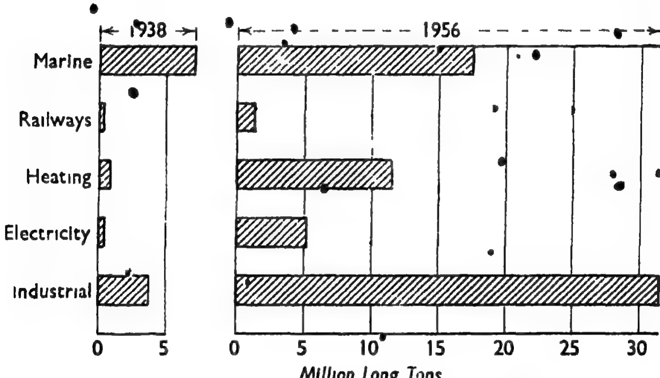
GRAPHS I—STATISTICAL

IN recent years there has been an increase in the use of diagrams of various kinds to help people to grasp facts and figures more readily, and to make deductions from these. For example, in the 1956 Report of the Bowater Paper Corporation there appeared the following diagrams to illustrate the number of people in the whole world and in several continents, and the number of those people who can read and write, i.e. who are literate. Such a diagram may be called a *column graph*. The widths of the columns are the same, and their lengths in this case represent the population of the world and of the various continents to a certain scale, which, in this case, was not stated but which was approximately 1 cm. to 300 million.



The diagrams show clearly that Asia is the continent with the largest number of illiterate people, that Africa is the most backward continent, that North America and Europe are the most advanced continents. Certain deductions can also be made fron the diagrams, e.g. that the population of Asia is approximately equal to that of all the others combined and that the population of Europe (including the U.S.S.R.) is much greater than that of North and South America combined.

The following column graph appeared in the 1956 Report of the Shell Transport and Trading Company to illustrate the growing use of oil for Power in Western Europe:



From a study of the graph:

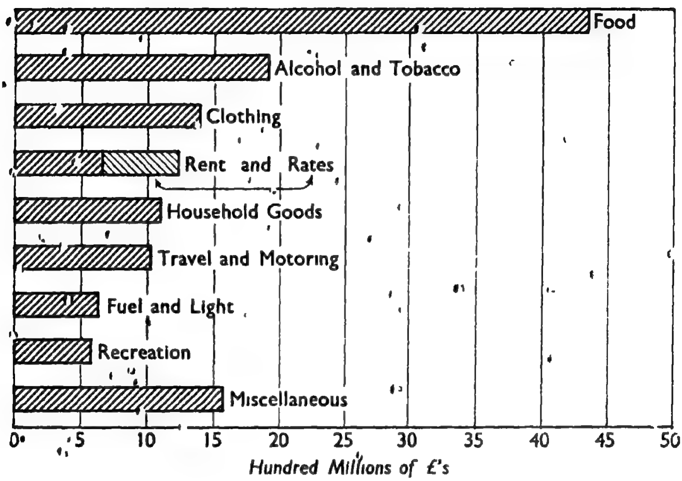
1. What is the scale?

2. Would you say that there has been a large or a small increase since 1938 in the use of oil by the Western European countries?

3. What is approximately the ratio of the amount of oil used in 1956 by ships to that used by the railways?

4. By estimating the quantity of oil consumed in the two years in each of the five cases, find in which case the rate of increase has been the greatest, and in which case the least.

The following graph was printed by the Corporation of the City of Glasgow and appears in a booklet giving a summary of the City's Finances:



From the column graph above:

1. What is the scale?

2. Estimate, in hundred millions of £s, the amount spent nationally on Food.

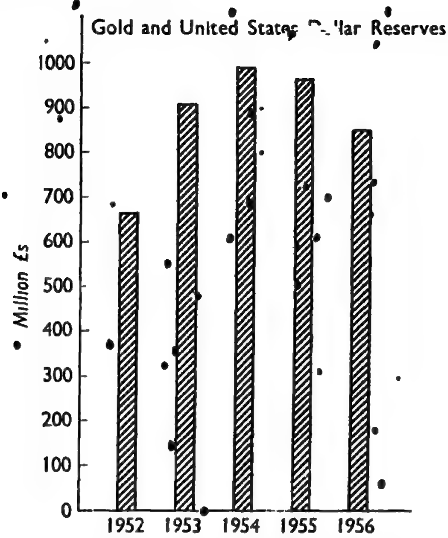
3. Similarly estimate the amount spent on Alcohol and Tobacco.

4 Estimate approximately the total national expenditure, and the percentage of the total paid in rent and rates.

5. Which is greater, the combined expenditure on alcohol and tobacco, travel and motoring, and recreation *or* that on rent and rates, household goods, and fuel and light?

In the column graphs considered so far, the columns run across the page, probably for convenience in printing; but

very often the columns run up and down the page, as in the following column graph:



1. What did the reserves stand at approximately in 1952, in 1954?

2. Between what years was the rate of increase greatest?

3. Between what years was the rate of decrease greatest?

The graph on p. 110 gives an estimate of the number of employees due to start and to stop work in a large city at certain times in the morning and evening.

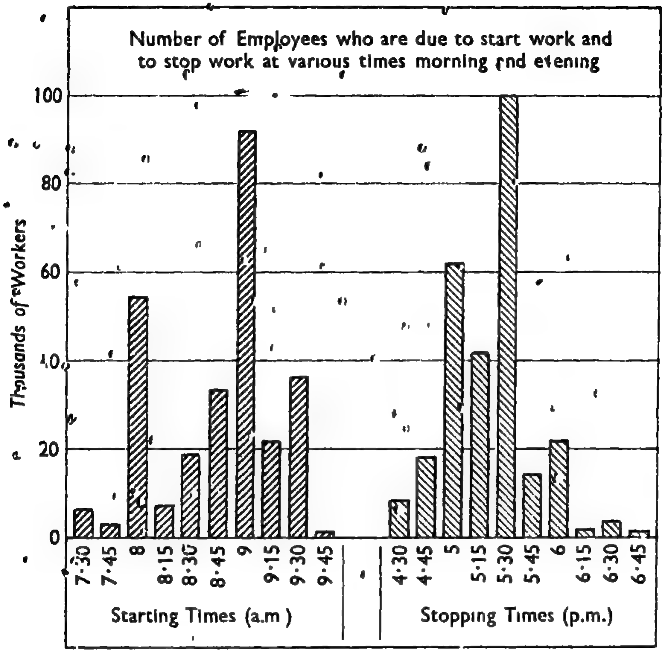
From the graph answer the following questions:

1. What two times of starting work appear to cause the greatest traffic congestion in the morning?

2. What two times of stopping work appear to cause the greatest traffic congestion in the evening?

3. When is the congestion likely to be worst (*a*) in the morning? (*b*) in the evening?

4. What suggestions would you make to employers to



reduce the traffic congestion: (*a*) in the morning; (*b*) in the evening?

Exercises 38

Construct column graphs to illustrate the following:

1. Daily Newspaper Circulation per 10,000 of the Population in 1952

Canada	240	India	10
U.S.A.	350	Sweden	490
United Kingdom	620	Japan	350
France	240	Australia	420

[Make width of rectangle $\frac{2}{10}$ in. and for lengths of rectangle let 1 in. represent 100.]

2. Population per Square Kilometre

U.S.A.	20	Haiti	116
Australia	1	Union of South Africa	11
New Zealand	8		
Canada	1	Nigeria	34

[Suggested scale: 1 in. represents 20.]

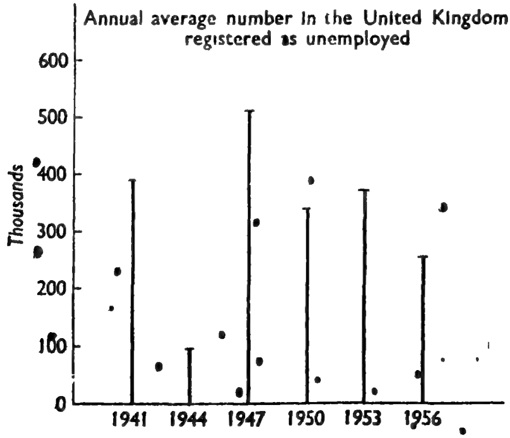
3. Life Expectancy at Birth, in Years

Canada	66	Norway	69
U.S.A.	62	United Kingdom	67
India	32	New Zealand	68
France	64		

4. Number of Inhabitants per Physician (1952)

Canada	900	New Zealand	700
U.S.A.	770	West Germany	750
Japan	1000	India	5700
United Kingdom	1200		

If, however, instead of having columns of the same width to represent the quantities being considered, we have straight lines, whose length, as in the graph, represent the quantities we would have a *Line Graph.*



From a study of the graph we can see at a glance the variation in the number unemployed from 1941 to 1956.

For example, it fell markedly between 1941 and 1944 and rose even more markedly between 1944 and 1947.

1944 was the best year and 1947 the worst year, as far as unemployment was concerned.

It is not possible from this graph to tell what the number of unemployed was in any year other than those marked, for we could not draw any line between those on the diagrams and obtain from it an estimate of the number unemployed in any particular year.

Exercises 39

Draw line graphs to illustrate the following:

1. Number (in Thousands to the Nearest Thousand) of Killed and Injured in Road Accidents from 1949 to 1956

1949	177	1953	227
1950	201	1954	238
1951	216	1955	268
1952	208	1956	268

[Suggested scale: 1 in. represents 50,000.]

2. Heights of Mountains (above Sea-level) in Feet. Correct to the Nearest 100 ft.

Mont Blanc	15,800	Kilimanjaro	19,300
Ben Nevis	4,400	Aconcagua	23,000
Mount Everest	29,000	Mount Cook	12,300

[Suggested scale: 1 in. represents 5000 ft.]

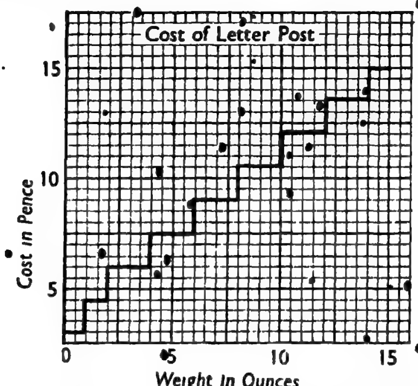
3. Electric Energy Production (in Thousand Million Kilowatt Hours)

Australia	12	United Kingdom	67
France	42	U.S.A.	514
Western Germany	61	Mexico	56
Norway	20	Japan	33

4. From the graph of the Cost of the Letter Post:

(i) What is the cost of sending letters with the following weights: $\frac{1}{2}$ oz., 3 oz., 7 oz., 11 oz.?

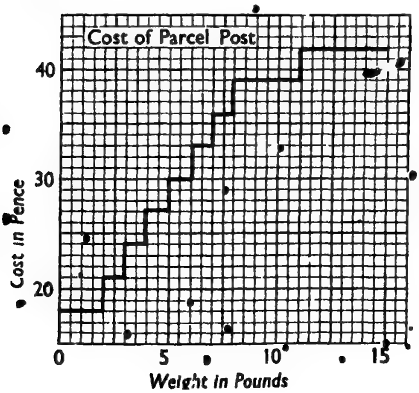
(ii) What is the heaviest letter that can be sent for $7\frac{1}{2}d.$, $10\frac{1}{2}d.$, $1s.$?

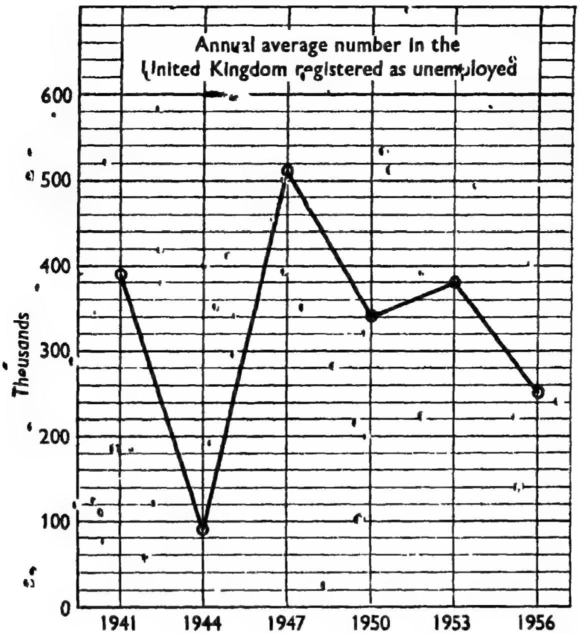


5. From the graph of the Cost of the Parcel Post:

(i) What is the cost of sending parcels with the following weights: 3 lb., 9 lb., $5\frac{1}{2}$ lb., 13 lb. 4 oz., $10\frac{1}{2}$ lb.?

(ii) What is the heaviest parcel that can be sent for $2s.\ 6d.$, $1s.\ 9d.$, $3s.\ 3d.$, $2s.\ 9d.$?



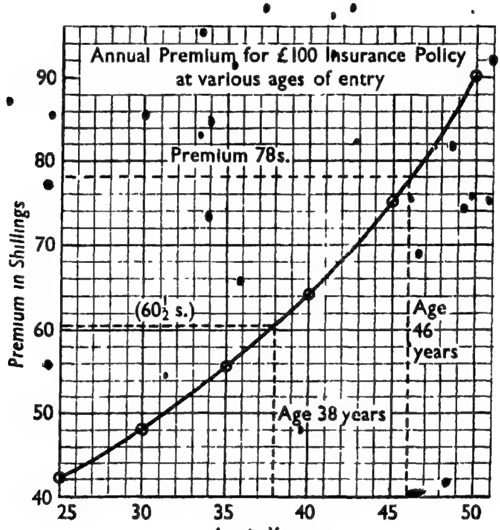


If we consider again the graph showing the numbers of unemployed in the United Kingdom, we might use squared paper as shown and instead of drawing straight lines as before to represent the numbers, we might mark only the top points of these lines. If further we join these top points by straight lines it would help us to see at a glance the changes in the number of unemployed in the various three-year periods. The rates of increase or decrease in these periods can be compared by comparing the slopes of the lines joining the points. For example, the rate of increase between 1950 and 1953 was less than that between 1944 and 1947. Similarly, the rate of decrease between 1941 and 1944 was greater than that between 1953 and 1956. As was

noticed before, of course, even when the points are joined by straight lines, we can deduce nothing about the figures of unemployed for any year other than those specified.

Consider now the following table, which gives the annual premium for £100 insurance policy:

Age at entry (in years) .	25	30	35	40	45	50
Premium (in shillings) .	42	48	$55\frac{1}{2}$	64	75	90



We draw on squared paper two straight lines at right angles as shown, marking ages in years on the line drawn across the page (scale 1 in. to 5 years), and marking the premium in shillings on the line drawn at right angles to this (scale 1 in. to 10 shillings). Since the earliest age mentioned is 25, this can be the first age marked in the graph, and similarly 40 is a convenient number to start with on the premium line.

The six points can now be fixed on the graph from the numbers given in the table above. These six points seem to lie on a smooth curve, which has been drawn, and the premium seems to increase with age according to a definite law. Hence, we can, from the graph, deduce the premium at any age between 25 and 50, not shown on the table, e.g. at age 38 the premium is $60\frac{1}{2}s$. Similarly, the age when a premium of 79*s*. is payable is 46 years.

In general, we may say that a graph is used to show in a pictorial way the relation between two varying quantities: in the previous graph these quantities are the age (in years) at entry, and the premium (in shillings). These quantities are called *variables*: one of them is called the *independent variable*, in this case the age; the other is called the *dependent variable*, in this case the premium, since the table shows the premiums at certain specified ages.

To draw the graph:

1. Two straight lines, called *axes*, are drawn at right angles as shown. The independent variable is usually marked on the axis across the page, while the dependent variable is marked on the one at right angles to this.

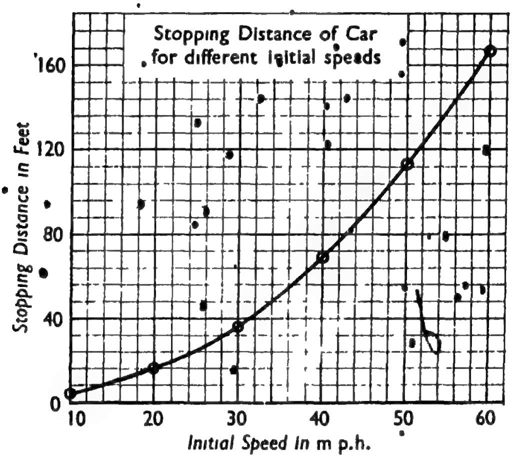
2. Suitable scales, generally as large as the squared paper will allow, are chosen for the two quantities and these are indicated clearly along the axes. Care should be taken in choosing the scale to render as easy as possible the plotting of points and the reading of results from the graph. Since the squared paper is usually marked in inches and tenths of an inch (or cm. and $\frac{1}{10}$ cm.) 1 in. may conveniently represent 1, 2, 3, etc., up to 10 units or any multiple of 10, e.g. 20, 50, 100, etc.

3. When the points are plotted these should be joined: (*a*) by a continuous line if the data for intermediate points can be inferred, e.g. the graph showing the premiums payable at certain ages for insurance; (*b*) by straight lines if

the data for intermediate points cannot be inferred, e.g. the number of unemployed in any particular year.

4. A suitable title should be put at the top of each graph.

Exercises 40



1. From the above graph estimate:

(*a*) the stopping distance if the initial speed is (i) 25 m.p.h., (ii) 45 m.p.h.;

(*b*) the maximum initial speed if the car is to stop within a distance of (i) 10 ft., (ii) 50 ft.

2. The following table gives the height of a stone, thrown vertically into the air, from the time it leaves the ground until it reaches the ground again:

Height in ft.	112	192	240	256	240	192	112	0
Time in sec.	1	2	3	4	5	6	7	8

Draw a graph, and from it find:

(i) when the stone reaches its highest point;
(ii) after how many seconds the stone is 156 ft. high;
(iii) the height of the stone after $4\frac{1}{2}$ sec.

3. The following table gives the number of miles per monthly ration of petrol for a certain car at constant speeds:

Monthly ration	Speed in m.p.h.	30	40	50	60	70	80
$10\frac{1}{2}$ gal.	No. of miles	321	315	292	256	218	181

Draw a graph and use it to find:

(i) the number of miles for a constant speed of 45 m.p.h.;

(ii) the constant speed so that the ration would suffice for 230 miles;

(iii) the constant speed if the car does 25 miles to the gallon.

4. The following table shows the height above the ground at intervals of one sec. of a body let fall from a height of 1000 ft.

Time in sec.	0	1	2	3	4	5	6	7
Height in ft.	1000	984	936	856	744	600	424	216

Find from a graph:

(i) the time when the body was 800 ft. high;

(ii) the height of the body after $5\frac{1}{2}$ sec.

5. The following table shows the height of the sea-level at a certain port during a complete day:

Time	12 mid-night	1 a.m.	2 a.m.	3 a.m.	4 a.m.	5 a.m.
Height of water	19′ 2″	16′ 4″	15′ 4″	17′ 1″	22′ 0″	25′ 10″
Time	6 a.m.	7 a.m.	8 a.m.	9 a.m.	10 a.m.	11 a.m.
Height of water	29′ 9″	31′ 10″	32′ 4″	30′ 6″	27′ 4″	24′ 0″
Time	12 noon	1 p.m.	2 p.m.	3 p.m.	4 p.m.	5 p.m.
Height of water	20′ 10″	18′ 3″	16′ 7″	16′ 7″	19′ 7″	23′ 5″
Time	6 p.m.	7 p.m.	8 p.m.	9 p.m.	10 p.m.	11 p.m.
Height of water	27′ 4″	30′ 7″	31′ 9″	31′ 2″	29′ 0″	25′ 10″

From a graph state:

(i) when it was high tide;

(ii) when it was low tide;

(iii) the approximate length of time between two high tides;

(iv) the time when the next low tide would occur.

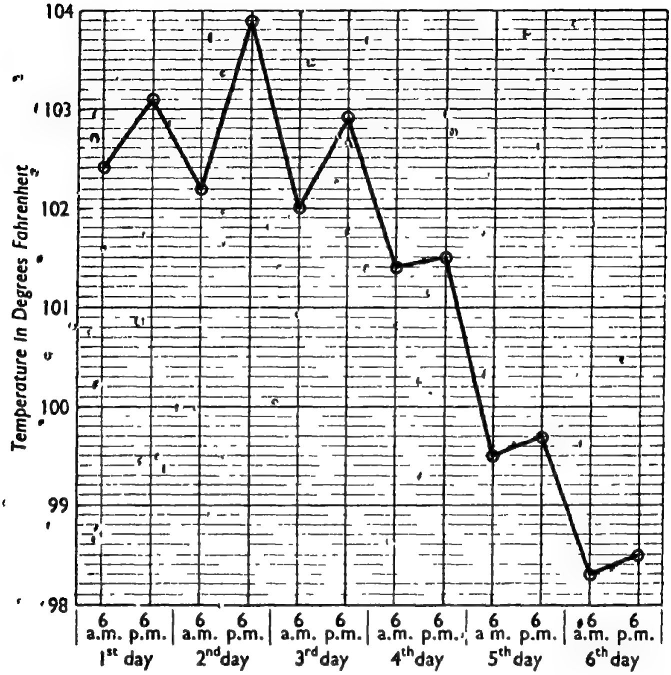
6. Barometer readings were taken every 2 hr. at a certain place during a day with the following results:

Time	8 a.m.	10 a.m.	12 noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.
Barometer reading in in.	29·30	29·18	29·25	29·35	29·37	29·46	29·52

From a graph estimate:

(i) the barometer readings at 9 a.m., 1.30 p.m., 5 p.m.;

(ii) the time when the barometer reading was 29·43 in.; 29·26 in.



7. The above graph shows the temperature record of a patient suffering from influenza. The readings were taken at 6 a.m. and 6 p.m.

(i) What scale is used to represent (*a*) temperature, (*b*) time?

(ii) What was the highest temperature reached?

(iii) Can you form any opinion as to the temperature at 12 noon each day? Why are the points joined by straight lines?

8. The monthly rainfall throughout a year at two places was as follows:

Place	Jan.	Feb.	Mar	Apr.	May	June
A	0·3	1·0	1·3	2·0	5·0	11·1
B	2·8	2·2	2·6	2·0	1·8	0·2
Place	**July**	**Aug.**	**Sept.**	**Oct.**	**Nov.**	**Dec.**
A	12·0	11·3	8·8	4·6	0·2	0·1
B	0·0	0·1	0·3	2·2	3·1	3·0

On the same page draw two graphs to illustrate the variation in the rainfall. From these graphs find

(i) Which place has wet summers and comparatively dry winters?

(ii) Which place has winter rains?

(iii) Which place has the greatest annual rainfall?

9. Draw a graph to show the poundage on Money Orders from the following data: Money Orders for sums up to £10 poundage 8*d.*; £20, 10*d.*; £30, 1*s.*; £40, 1*s.* 2*d.*; £50 (limit), 1*s.* 4*d.*

10. The height of the barometer at various altitudes is as follows:

Height (ft.)	0	1000	2000	3000	4000	6000	8000	10,000
Barometer reading (in. of mercury)	29·9	28·9	27·8	26·8	25·8	24·0	22·2	20·6

From a graph estimate:

(i) the reading of the barometer at 3500 ft. and 9000 ft.;

(ii) the height at which the barometer reading will be 23·1 in., 26·0 in.

11. The extension of a wire under certain loads is as follows:

Load in lb.	15	20	25	30	35	40
Extension in in.	0·5	1	2·4	7·6	17·6	30

From a graph estimate:

(i) the extension of this wire for a load of 28 lb.;
(ii) the load that would produce an extension of 20 in.

12. The following table shows the length of a pendulum for various periods:

Period in sec	0·25	0·5	0·75	1	1·25	1·5	1·75	2
Length in in.	0·6	2·4	5·5	9·8	15·3	22·0	30·0	39·2

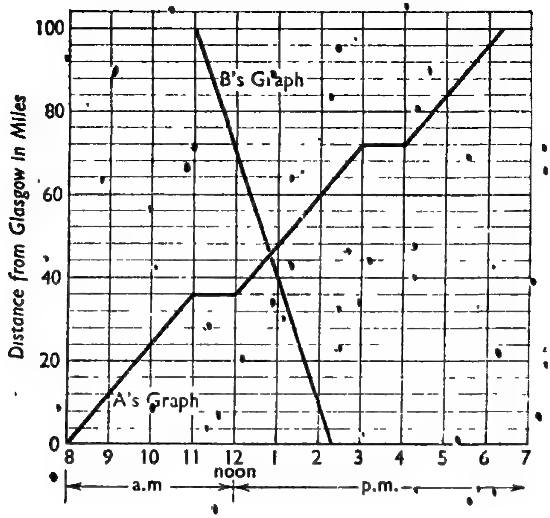
From a graph:

(i) find the period of a pendulum $1\frac{1}{2}$ ft. long;
(ii) find the length of a pendulum of period 1·6 sec.

Exercises 41

1. Two men A and B travel on the Glasgow–Carlisle road. A travels from Glasgow to Carlisle, B from Carlisle to Glasgow. The graph on p. 123 gives their distances from Glasgow at any time.

(i) What is the distance between Carlisle and Glasgow?
(ii) When did A and B start on their journeys?
(iii) At what rate are they travelling?
(iv) Which of them rested and when did he rest?
(v) When did they pass each other and at what distance from Glasgow?
(vi) When were they 20 miles apart?
(vii) When did they arrive at their destinations?



2. Two places A and B are 90 miles apart. A non-stop train leaves A for B at 9 a.m. A non-stop train leaves B for A at 9.30 a.m. The first train arrives at B at 11.15 a.m., and the second train arrives at A at 11.30 a.m. When do the trains pass each other, and at what distance from A?

3. Two towns A and B are 120 miles apart. A non-stop train travelling at 40 m.p.h. leaves A for B at 8 a.m. A second train travelling non-stop at 35 m.p.h. leaves B for A at 9 a.m. When will the trains pass each other, and how far from B?

4. One man walking at $3\frac{1}{2}$ m.p.h. sets out at noon for a certain place, and a second man walking at 4 m.p.h. sets out for the same place $\frac{1}{2}$ hr. later. When will this second man overtake the first?

5. If in Example 4 the first man had rested for $\frac{1}{4}$ hr. after each hour's walking, and the second man had not rested at all, find when he would now have overtaken the first man.

6. At 10 a.m. a man sets out to walk to a town 20 miles distant. He walks at 4 m.p.h. and rests for 15 min. after each hour's walking. At 12 noon a cyclist sets out from the same place to cycle to the same town. He cycles there at 10 m.p.h., waits 15 min. in the town and cycles back at 12 m.p.h. When will he pass the walker on the outward, and on the return journey?

7. A tank holds 100 gal. It can be filled by two taps, A and B, A delivering 5 gal. per min., B 8 gal. per min. When the tank is empty A is turned on alone for 4 min. 24 sec. B is then turned on and both run until the tank is full.

Draw a graph to show the amount of water in the tank at any time from the beginning until the tank is full. Find from the graph:

(i) how long it takes to fill the tank;

(ii) how long it is before the tank is half full;

(iii) how much water is in the tank 8 min. from the start.

REVISION PAPERS 16–20

Paper 16

1. Simplify $5x - [3y - 2\{x - y - 3(2 - x)\}]$, and find the value of the answer when $x = -3$, $y = -2$.

2. Simplify:

$$\text{(i)} \quad \frac{3a^3}{2b^2} \times \frac{-b}{a^2} \div \frac{b^2}{(-2a)^2}$$

$$\text{(ii)} \quad \frac{3x}{4} - \frac{x+2}{3} + \frac{5}{6}$$

3. Solve $x - \dfrac{x-3}{4} + 1 = 0.$

If in addition $3x - y = 7$, find y.

4. If $ax + 2y + 3 = 0$, and $x = 1$, when $y = -2$, find a. Then using this value of a in the equation, find the value of y when $x = -1$.

5. A straight line is divided into two parts, the first being $(3x - 4)$ in. long, and the other $(5x - 2)$ in. long. If the first part is one-third of the length of the whole line, find the length of the whole line in inches.

6. A man spends £5 in buying a fountain pens. He sells them at a fixed price, and the sale of b of the pens brings him his £5. If he sells the remaining pens at the same fixed price, find an expression for his total profit in pounds.

Paper 17

1. A man bought x books at a shillings each. He sold half of them at b shillings each, and the rest at a sale, when he reduced his selling price by 50%. His total gain was £p. Find a formula for p. Find his gain when $x = 120$, $a = 6$, $b = 10$.

2. Simplify:

(i) $-72x^2yz \div 9x^3y^3z^2 \times -2x^2y$

(ii) $\frac{3}{8x} - \frac{4}{3x} + \frac{5}{6x}$

3. Solve $3(y - 3) = 2(2y - 5) - 4(\frac{3}{4} - y)$.

4. Find the values of $\frac{(x-2)(5-x)}{4}$ for the following values of x: $-2, -1, 0, \frac{1}{2}, 3$.

5. The average age of a class of n boys is y years. 5 boys, each aged 12 years, leave the class and 4 boys, each aged k years, join the class. If the average age of the class is still y years, find a formula for k.

6. A motor car does x miles to the gallon of petrol. Its owner sets out to visit three towns A, B and C in succession. At the start of his journey the mileometer reading is 36,145 and the petrol tank contains 12 gal. At A, B, C the readings for the petrol tank are respectively $10\frac{1}{2}$, 7, $4\frac{1}{4}$, and at C the mileometer reads 36,362. Find x and the distance of B from C.

Paper 18

1. If l lb. of tea cost s shillings:

(i) How much tea will I get for £2?

(ii) How much will a lb. tea cost?

2. Solve $\frac{1}{2}\left(\frac{x}{2} - 3\right) - 3\left(1 - \frac{x}{2}\right) = \frac{2x}{3} - 1\frac{1}{4}$.

3. Multiply $x^2 - xy + y^2$ by $-xy^2$ and also by $-x^2y$ and add the results. What is the value of the answer when $x = y = -1$?

4. If $\frac{x^2 + y^2}{x^2 - xy + y^2} = 5a$ when $x = -3$, $y = -4$, find a and prove that $\frac{1}{4}(1 - a) - \frac{2}{13} = 0$.

5. What number falls short of 45 by as much as 91 exceeds three times itself?

6. An express train travelling at an average speed of v m.p.h. can cover a journey of 400 miles according to time-table. If on one journey it arrived 10 min. early, find an expression for its average speed on that journey. Do not try to simplify the expression.

Paper 19

1. A man works a 44-hr. week. His wage-rate is s shillings per hour, and he is paid double time for overtime. If in one week he has $1\frac{1}{2}s$ hr. overtime, find how many pounds he earns that week.

2. A father is 3 times as old as his son. 10 years ago he was 13 times as old as his son. Find the father's present age.

3. Solve,
$$\frac{1}{x} - 4 = \frac{5}{y}$$
$$\frac{3}{y} = 2 - \frac{6}{x}$$

4. A certain expression when divided by $-3x$ gives a quotient of $-2x^2y + 3xy^2 + 4y^3$ and a remainder -48. Find the expression, and find its value when $x = 1, y = -2$.

5. (i) If $l = \sqrt{6Rh - 3h^2}$, find l if $R = 33{\cdot}8$, $h = \frac{1}{10}$.

(ii) If $\frac{1}{v} - \frac{1}{u} = \frac{2}{r}$, find v if $u = -5$, $r = 20$.

6. A thin metal wire forms the outline of a rectangular prism $(3x - 2)$ in. long, $(8 - x)$ in. broad, $2x$ in. high. Find the total length of the wire. If its length is 6 ft., find the dimensions of the prism.

Paper 20

1. (i) A man sold a car for £x and thereby gained 15% of what it cost him. What did it cost him?

(ii) How many minutes have elapsed between noon and k min. to 4 p.m.?

2. A man's wages are s shillings per hour for a working week of x hr. He is paid y shillings per hour for overtime. How many hours overtime did he work in the week when his wages were £$3s + s$ shillings. Check your result when $s = 8$, $y = 12$, $x = 40$.

3. Solve:

(a) $4{\cdot}4x - 2{\cdot}3 = 1{\cdot}5 - 3{\cdot}2x$
(b) $0{\cdot}3(x - 0{\cdot}4) + 1{\cdot}5(2x - 0{\cdot}4) \quad 2{\cdot}5(x - 1)$

4. The equation $\frac{2}{3}\left(2a + \frac{x}{3}\right) + 4\frac{1}{4} = 3\left(a + \frac{x}{2}\right)$ is satisfied when $x = -18$. Find the value of a.

5. If $x = 4\left(\frac{a}{2} - b\right) \quad 2b$, $y = 3(a - 3b) - (3a + 5b)$, $z = a - 4(a + b)$, find the value of $3x \quad y + 2z$ in terms of a and b.

6. A man walks at 3 m.p.h. and runs at 6 m.p.h. He covers a mile, partly by walking and partly by running, in 15 min. For how long does he walk? How far does he run?

CHAPTER 15

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE WITH TWO UNKNOWNS

IN Chapter 11 it has been found that one and only one value of x will satisfy an equation such as $\frac{3}{4}x - 7 = 8$, namely the value 20.

If the equation $x + y = 8$ is considered, it will be found than an unlimited number of pairs of values of x and y can be obtained that will satisfy the equation. A few of these pairs are shown below:

x	-2	$-\frac{1}{2}$	0	$1\frac{1}{2}$	3	5	6	9
y	10	$8\frac{1}{2}$	8	$6\frac{1}{2}$	5	3	2	-1

Similarly, if the equation $x - y = 2$ is considered, an unlimited number of pairs of values of x and y can be obtained that will satisfy the equation. A few of these pairs are shown below:

x	-2	$-\frac{1}{2}$	0	$1\frac{1}{2}$	3	5	6	9
y	4	$-2\frac{1}{2}$	-2	$-\frac{1}{2}$	1	3	4	7

On examining the two tables it will be noticed that $x = 5$, $y = 3$ is a solution of both equations, and there is no other common solution.

$x = 5, y = 3$ is said to be the solution of the simultaneous equations $x + y = 8$ and $x - y = 2$. [*Simultaneous*, because they are to be true *at the same time*.]

F

It would be a difficult and tedious business, if the above method of trial and error was the only method of solving simultaneous equations such as those we have considered. There are, however, two general methods that can be used, and these are shown below:

Example: Solve:

$$2x - 3y = 5 \qquad (1)$$
$$3x + 2y = 14 \qquad (2)$$

First Method—Substitution—

$$\text{From (1), } x = \frac{3y + 5}{2}$$

Substitute this value for x in (2).
[This gives an equation with no term in x.]

$$3\left(\frac{3y + 5}{2}\right) + 2y = 14$$
$$\therefore \quad 3(3y + 5) + 4y = 28$$
$$\therefore \quad 9y + 15 + 4y = 28$$
$$\therefore \quad 13y = 13$$
$$y = 1$$
$$\text{But } x = \frac{3y + 5}{2}$$
$$\therefore \quad x = \frac{3(1) + 5}{2}$$
$$\text{i.e. } x = 4$$

Solution is

x	4
y	1

Note: In using the Substitution Method, we can substitute the value of x or of y, obtained from either equation, in the other. A little thought beforehand will sometimes show that one substitution is to be preferred to the others. For example, in solving the two equations:

$$3x + 7y = 15 \qquad (1)$$
$$4x + y = 20 \qquad (2)$$

by this method, it would seem best to find the value of y from equation (2), $y = 20 - 4x$, and substitute this in equation (1) thus:

$$\begin{aligned} 3x + 7(20 - 4x) &= 15 \\ \therefore\ 3x + 140 - 28x &= 15 \\ \therefore\ -25x &= -125 \\ x &= 5 \\ \text{and } y &= 20 - 4x \\ &= 20 - 20 \\ &= 0 \end{aligned}$$

Had we chosen any of the other substitutions, e.g. the value of x in terms of y from equation (1); namely $x = \frac{15 - 7y}{3}$, it would have resulted in an equation containing fractions, and the working would have been longer. When all possible substitutions involve fractions, it is advisable to choose the one which involves the smallest denominator.

Second Method—Elimination—

$$2x - 3y = 5 \qquad (1)$$
$$3x + 2y = 14 \qquad (2)$$

Multiply (1) by 2 and (2) by 3 and add

[This is done to eliminate the terms in y.]

$$\begin{aligned} \therefore\ 4x - 6y &= 10 \\ \text{and } 9x + 6y &= 42 \\ \text{Adding} \therefore\ 13x &= 52 \\ \therefore\ x &= 4 \end{aligned}$$

Substitute 4 for x in (2).

$$\begin{aligned} \therefore\ 3(4) + 2y &= 14 \\ \therefore\ 12 + 2y &= 14 \\ \therefore\ 2y &= 2 \\ \therefore\ y &= 1 \end{aligned}$$

Solution is

Note: The substitution of 4 for x could be made in either (1) or (2), and in this case it would not matter much which was chosen. In general, the substitution should be made in the equation which will give the answer more readily.

In the Elimination Method, we form, from the two given equations, two equations with the numerical coefficients of either x or y the same. Then by addition or subtraction we eliminate one of the unknowns. Careful consideration should be given to the question as to which unknown is the easier to eliminate. For example, in solving the two equations

$$3x - 11y = 17 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$9x + 13y = 5 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

it would seem best to eliminate x by multiplying (1) by 3 and subtracting it from (2).

$$9x + 13y = 5$$
$$9x - 33y = 51$$
$$46y = -46, \text{ etc.}$$

Exercises 42

In Exercises 1–6, find the value of y, in terms of x:

1. $x + y = 5$ 3. $x + 3y = 7$ 5. $2x - 3y + 5 = 0$
2. $x - y = 4$ 4. $3x - 2y = 5$ 6. $4y + 2x - 1 = 0$

In Exercises 7–12, find the value of x, in terms of y:

7. $x + y = 3$ 9. $2x - 3y = 4$ 11. $3x + 4y - 7 = 0$
8. $5y - 3x = 0$ 10. $y - 5x = 7$ 12. $3y - 2x + 8 = 0$

Solve, and check the solutions:

13. $x + y = 7$; $x - y = 3$ 14. $3x + y = 5$; $x + y = 3$

15. $x + 2y = 9$
$x - y = 0$

16. $x + 4y = 9$
$3x - 4y = -5$

17. $2x + 3y = 13$
$3x - 2y = 0$

18. $3x + 4y = 1$
$5x - 2y = -7$

19. $x - y = 1$
$3x = 2y$

20. $5x - y = 9$
$y - 2x = 3$

21. $7x + 2y = -3$
$3x - 5y = 28$

22. $4x + 3y = 1$
$6x - 5y = -8$

23. $3x + 4y - 3 = 0$
$6y = 9x$

24. $4x + 2y - 11 = 0$
$3x = y + 7$

Solve:

25. $7x + 3y = 6$
$5y - 9x = 10$

26. $6x + 5y = 11$
$5x - 2y = 3$

27. $13x + 15y = 19$
$7x + 9y = 13$

28. $5x + 4y = 40$
$9x - 7y = 1$

29. $7x - 11y + 21 = 0$
$3y - 5x = 15$

30. $8y - 7x + 9 = 0$
$11x + 3y + 17 = 0$

31. $6x + 3y = 11$
$5x + 2y = 8$

32. $4x - 5y = -6$
$24x = 9y - 1$

33. $3x - 5y = 5x + 3y = 16$

34. $5x - 2y = 9x - 5y = 7$

35. $5x + 3y - 2 = 2x + 7y = 11$

36. $8x + 5y + 5 = 3y + 7x + 3 = 0$

37. $4x - 3y = 11$
$3(x + 2) = 7y$

38. $3x - 1 = 4(y + 1)$
$7(2x + 3) - 6y = 0$

39. $3(x - 7y) = -4$
$9(3 - x) = 5(y + 3)$

40. $3(x - 3y) = 5(3y - x)$
$2(3x - y) = 3(4y + x) + 5$

41. $3(x + y) + 2(y - x) = 4$
$2(x - 2) - 3(y - 3) = 0$

42. $3(2x + y) + 4 = 2(3y - 4x - 1)$
$4(x + 2) = 3(y + 3) - 2$

43. $x + y = 5a$
$x - y = a$

44. $2x + 3y = -7a$
$3x - y = 6a$

45. $2x + y = 3a + b$
$x - 2y = 3b - a$

46. $ax + 3y = 6a$
$2ax - 5y = a$

47. $ax + by = ac$
$bx + cy = bc$

48. $2x - 3y = 6x + 5y - 7a$

49. $x + y = c$
$3x - 2y = d$

50. $ax - by = b$
$bx - ay = a$

Simultaneous Equations Involving Fractions

Example 1: Solve:

$$\tfrac{3}{8}x - \tfrac{1}{4}y = 1 \qquad (1)$$
$$\tfrac{1}{2}x + 3y = 8 \qquad (2)$$

Clear both equations of fractions, by multiplying (1) by 8, and (2) by 2,

$$\therefore\ 3x - 2y = 8 \qquad (3)$$
$$x + 6y = 16 \qquad (4)$$

Multiply (3) by 3 and add it to (4),

$$9x - 6y = 24$$
$$x + 6y = 16$$
$$\therefore\ 10x = 40$$
$$\therefore\ x = 4$$

Substitute 4 for x in (4),

$$\therefore\ 4 + 6y = 16$$
$$\therefore\ 6y = 12$$
$$\therefore\ y = 2$$

Solution is

x	4
y	2

Example 2: Solve:

$$\frac{4}{x} + \frac{3}{y} = -5 \qquad (1)$$

$$\frac{3}{x} - \frac{2}{y} = 4\tfrac{3}{4} \qquad (2)$$

These equations may be rewritten thus

$$4 \times \frac{1}{x} + 3 \times \frac{1}{y} = -5 \qquad (1)$$

$$3 \times \frac{1}{x} - 2 \times \frac{1}{y} = 4\tfrac{3}{4} \qquad (2)$$

Put p for $\frac{1}{x}$, and q for $\frac{1}{y}$ in (1) and (2).

The equations become

$$4p + 3q = -5$$

$$3p - 2q = 4\tfrac{3}{4}$$

Solving these two equations, $p = \frac{1}{4}$, $q = -2$

$$\text{Now } \frac{1}{x} = p \text{ and } \frac{1}{y} = q$$

$$\therefore\ x = \frac{1}{p} \text{ and } y = \frac{1}{q}$$

$$\therefore\ x = 4,\ y = -\tfrac{1}{2}$$

Solution is

x	4
y	$-\frac{1}{2}$

Exercises 43

Solve the following equations:

1. $x - \dfrac{y}{2} = 2$
   $3x + \dfrac{y}{4} = 13$

2. $\dfrac{x}{2} - 2y = 1$
   $\dfrac{x}{3} + y = 3$

3. $\frac{x}{3} + \frac{y}{2} = 2$
$2x - 3y = 0$

4. $\frac{x}{2} + \frac{y}{4} = 5$
$\frac{x}{3} - \frac{y}{2} = 6$

5. $\frac{x}{3} + \frac{y}{5} = 1\frac{1}{2}$
$\frac{3x}{4} - \frac{y}{2} = 1$

6. $\frac{2}{3}x - \frac{5}{6}y = 1\frac{1}{2}$
$3x - 4y = 8$

7. $\frac{1}{2}x + \frac{2}{3}y = \frac{1}{6}$
$\frac{1}{4}x + \frac{1}{2}y = \frac{1}{4}$

8. $\frac{3}{5}x - \frac{1}{4}y = 8$
$\frac{2}{3}x + y = 14$

9. $\frac{1}{8}(u + v) = \frac{1}{3}$
$\frac{1}{4}u - \frac{3}{5}v = 1\frac{4}{5}$

10. $\frac{2}{3}x - \frac{1}{6}y + \frac{11}{24} = 0$
$\frac{1}{4}x + \frac{1}{3}y - \frac{1}{8} = 0$

11. $\frac{x + y}{3} - 2 = 2x$
$1 - \frac{x - y}{2} = 2y$

12. $\frac{2x}{3} - \frac{3x + 2y}{4} = \frac{2}{3}$
$\frac{2x + 3y}{5} + \frac{x}{2} = 3\frac{1}{5}$

13. $3(p + 2) = 4(q - 3)$
$\frac{3q + 4}{2} - \frac{p - 5}{6} = 0$

14. $\frac{1}{3}(p + 9) - \frac{1}{2}(q - 10) = p + 2\frac{1}{2}q = 4$

15. $\frac{x}{3} + \frac{y}{4} - 1\frac{1}{2} = \frac{7x}{12} - \frac{y + 12}{8} = 0$

16. $\frac{2}{5}x - \frac{1}{3}y = \frac{9}{10}x - y = \frac{1}{10}$

17. $\frac{1}{4}x = 2 + y$
$\frac{1}{2}(2x + y) = \frac{3}{\text{[illegible]}}(\text{[illegible]} - y)$

18. $0{\cdot}3\,v - 0{\cdot}2u = 3{\cdot}2$
$0{\cdot}5\,v - 0{\cdot}3u = 5$

19. $0{\cdot}3x + 0{\cdot}1y = 1{\cdot}2$
$2x = 5y - 5{\cdot}6$

20. $\frac{x}{0{\cdot}2} - \frac{y}{0{\cdot}3} = 11$
$10x + y + 1 = 0$

21. $\frac{x}{2} - \frac{y}{5} = \frac{2x}{3} - \frac{3y}{10} = 2x - y - 2$

22. $\frac{1}{3}(2x - 3y + 10)$
$= \frac{1}{4}(y + 2x + 1)$
$= 2(y - x) - 3$

23. $0{\cdot}3x + 0{\cdot}5y = 1{\cdot}5$
$0{\cdot}4x - 0{\cdot}2y = 0{\cdot}7$

24. $\dfrac{x}{0{\cdot}2} + 0{\cdot}2y = 10{\cdot}2$
$0{\cdot}3x + \dfrac{y}{0{\cdot}2} = -4{\cdot}4$

25. $0{\cdot}4x + 0{\cdot}3y = 0{\cdot}45$
$1{\cdot}6x + 0{\cdot}4y + 1 = 0$

26. $\dfrac{1}{x} + \dfrac{1}{y} = 3$
$\dfrac{1}{x} - \dfrac{1}{y} = 1$

27. $\dfrac{6}{x} - \dfrac{1}{y} = 5\frac{1}{2}$
$\dfrac{3}{x} + \dfrac{2}{y} + 1 = 0$

28. $5u - \dfrac{3}{v} = 5$
$7u + \dfrac{2}{v} = 0{\cdot}8$

29. $\dfrac{5}{x} - \dfrac{2}{y} = \dfrac{3}{x} + \dfrac{4}{y} + 8 = -6$

30. $\dfrac{3}{x} - \dfrac{2}{y} = 1$
$\dfrac{2}{x} + \dfrac{5}{y} - 26 = 0$

31. $\dfrac{y}{b} + \dfrac{x}{a} = \dfrac{1}{b}$
$\dfrac{y}{a} - \dfrac{x}{b} = \dfrac{1}{a}$

32. $\dfrac{x}{a} + \dfrac{y}{b} = -\dfrac{b}{a}$
$\dfrac{x}{2a} + \dfrac{2y}{b} = \dfrac{2b}{a}$

Chapter 16

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE IN TWO UNKNOWNS

Example 1: 5 lb. tea and 3 lb. coffee cost altogether £2 14s., while 3 lb. tea and 5 lb. coffee cost altogether £2 18s. Find the price of 1 lb. of each.

Let x shillings = price of 1 lb. tea

y shillings = price of 1 lb coffee

Cost of 5 lb. tea and 3 lb. coffee = $(5x + 3y)$ shillings

" 3 " 5 " = $(3x + 5y)$ "

$\therefore$ $(5x + 3y)$ shillings = £2 14s

$\therefore$ $(5x + 3y)$ shillings = 54 shillings

$$\therefore 5x + 3y = 54 \quad (1)$$

$$\text{Similarly, } 3x + 5y = 58 \quad (2)$$

Solving these two simultaneous equations,

$$x = 6, \ y = 8$$

$\therefore$ Cost of 1 lb. tea = 6s.

" 1 lb. coffee = 8s

Example 2: The value of two heaps of coins, one of half-crowns, the other of florins, is £3 17s The value would be decreased by 1s, if the number of half-crowns was doubled and the number of florins halved. Find the original number of half-crowns, and of florins.

Let x = original number of half-crowns

Let y = " " florins

$\therefore$ x half-crowns + y florins = £3 17s.

$\therefore$ $5x$ sixpences + $4y$ sixpences = 154 sixpences

$$\therefore 5x + 4y = 154 \quad . \quad . \quad . \quad (1)$$

If the number of half-crowns was doubled, and the number of shillings halved, we would have

$$2x \text{ half-crowns and } \tfrac{1}{2}y \text{ florins}$$
$$\text{New value} = £3\ 17s. - 1s. = £3\ 16s.$$
$$\therefore\ 2x \text{ half-crowns} + \tfrac{1}{2}y \text{ florins} = £3\ 16s.$$
$$\therefore\ 10x \text{ sixpences} + 2y \text{ sixpences} = 152 \text{ sixpences}$$
$$\therefore\ 10x + 2y = 152 \quad . \quad . \quad (2)$$

Solving equations (1) and (2), $x = 10$, $y = 26$

$\therefore$ Original number of half-crowns is 10.

" " florins is 26.

Example 3: A certain number of two digits is increased by 27, if the digits are reversed. The units digit of the original number is twice the tens digit. Find the original number.

Let x be the tens digit

and y be the units digit

The units digit is twice the tens digit

$$\therefore\ y = 2x \quad . \quad . \quad (1)$$

The number is $10x + y$. When the digits are reversed the new number is $10y + x$.

$$\text{New number} = \text{Old number} + 27$$
$$\therefore\ 10y + x = 10x + y + 27$$
$$\therefore\ 9y - 9x = 27$$
$$\therefore\ y - x = 3 \quad . \quad . \quad . \quad (2)$$

Solving equations (1) and (2), $x = 3$, $y = 6$

$\therefore$ Original number is 36.

Exercises 44

1. The sum of two numbers is 37 and their difference is 13. Find them.

2. Two numbers are such that one-fifth of their sum is 12 and one quarter of their difference is 2. Find them.

3. Two numbers are such that twice the smaller added to three times the larger is 37, while the difference between the numbers is $\frac{2}{7}$ of their sum. Find the numbers.

4. Find two numbers such that if 7 is added to the greater the answer is twice the smaller number, while if 14 is added to the smaller it is then equal to the greater.

5. Find two numbers such that four times the smaller exceeds twice the larger by 14, and three times the larger exceeds four times the smaller by 5.

6. Three numbers are such that the middle one is equal to the average of the other two. Their sum is 33, and the largest exceeds the smallest by 12. Find the numbers.

7. The sum of two fractions is eight times their difference. Three times the smaller exceeds the larger by 1. Find the fractions.

8. Four times the average of two numbers is equal to three times the larger number, while the difference between the numbers is three less than the average of the numbers. Find the numbers.

9. Two numbers are such that the sum of the first and half of the second is 22, while the sum of the second and half of the first is 26. What are the numbers?

10. Two numbers are such that the greater of the two is 4 less than twice the smaller. If the greater is exactly 3 times the difference between the numbers, find the numbers.

11. The sum of the ages of a father and son is 79 years. The difference in their ages is 29 years. Find their ages.

12. In 10 years' time a son will be as old as his father was 15 years ago. Their present ages total 61 years. Find their present ages.

13. Twenty years ago, A was twice as old as B, and, in 10 years time, he will be $1\frac{1}{2}$ times as old. Find their present ages.

14. In 20 years' time a father will be exactly twice his son's age. When the son is as old as the father is now, the sum of their ages will be 85 years. What are their present ages?

15. When his son was born, a father's age was six times his son's present age. When the son reaches his father's present

age, the father will be 52 years old. What are their present ages?

16. A father has two sons whose ages differ by 3 years. In 20 years' time the sum of the sons' ages will equal the father's age. Find the present age of the younger son, if, when he was born, the father was nine times as old as his elder son.

17. If the numerator and denominator of a certain fraction are each increased by 1 the fraction is equal to $\frac{1}{2}$. If they are each decreased by 1 the fraction is equal to $\frac{1}{4}$. Find the fraction.

18. The denominator of a fraction exceeds the numerator by 4. If 3 is added to both the numerator and the denominator the fraction is equal to $\frac{2}{3}$. Find the fraction.

19. If the denominator is subtracted from the numerator of a certain fraction, and 1 added to the denominator, the value of the fraction so formed is 1. The sum of the numerator and denominator of the original fraction is 16. Find the fraction.

20. The denominator of a fraction is greater by 1 than 4 times the numerator. When the numerator is multiplied by 3 and the denominator increased by 3, the value of the fraction is $\frac{1}{2}$. Find the fraction.

21. When the numerator and denominator of a fraction are each increased by 1 the fraction has the value $\frac{1}{2}$. If the denominator is doubled and the numerator increased by 3 the value of the fraction is unaltered. Find the fraction.

---

22. A number consists of two digits whose sum is 7. If the number, when increased by 3, is equal to eleven times the tens digit, find the number.

23. A number consists of two digits whose sum is 13. When the digits are reversed the number is decreased by 45. Find the number

24. A number consists of two digits whose sum is 9. When the digits are reversed the number is increased by 45. Find the number.

25. The tens digit of a two-digit number is half the units digit. When the digits are reversed the number is increased by 27. Find the number.

26. A two-digit number exceeds 5 times the sum of its digits by 6. If increased by 6 the number is 12 times the tens digit. Find the number.

27. A number consists of two digits. When the number is added to the number formed by reversing the digits the result is 143, while the number itself is 2 less than 6 times the sum of its digits. What is the number?

---

28. In a heap of 58 coins there are two kinds only—florins and half-crowns. The total value is £6 11*s*. Find how many coins there are of each kind.

29. A sum of £3 10*s*. is made up of half-crowns and shillings. If the numbers of half-crowns and shillings were interchanged the total value would diminish by one guinea. How many coins of each kind are there?

30. A sum of two guineas is composed of shillings and threepences. If the numbers of shillings and threepences are interchanged the sum is decreased by 9*s*. How many coins of each kind are there?

31. A collection consists of half-crowns, shillings and sixpences. There are three times as many shillings as there are half-crowns. The value of the collection is £8 10*s*. and there are 200 coins altogether. How many coins of each kind are there?

32. A bag contains half-crowns, shillings and sixpences, and there are twice as many shillings as there are half-crowns. If all the coins had been shillings, the total sum of money in the bag would not be altered, but, if all the coins had been six-

pences, the total value would be £1 10*s.* less. How much money was in the bag?

33. 5 lb. tea and 8 lb. coffee cost £4 6*s.*; and 9 lb. tea and 6 lb. coffee cost £4 16*s.* Find the cost of tea and of coffee per lb.

34. A family of two adults and two children go on an outing which costs £2 8*s.* The charge for a mother and three children on the same outing is £2 2*s.* Find the cost for each adult and each child.

35. A man wishes to save exactly £100 in a year. He saves £2 10*s.* each week and wishes to reduce this to £1 10*s.* each week as soon as possible. After how many weeks can he do this?

36. A load of fuel consisting of 5 tons of coal and 2 tons of coke is bought for £40. If a load containing half this amount of coal and twice this amount of coke can be bought for £35, what is the price of 1 ton of coal and of 1 ton of coke?

37. Fifteen shillings was spent in buying 50 oranges some at 3 for 1*s.*, others at 4 for 1*s.* How many of each kind were bought?

38. The subscriptions to a tennis club were £5 5*s.* for adults and £2 2*s.* for juniors. The total subscriptions amounted to £504, and there were 120 members altogether. How many of each kind were there in the club?

39. 350 tickets were sold for a concert. The tickets were priced at 4*s.* 6*d.* and 2*s.* 6*d.* each and the amount received for the dearer seats was £1 15*s.* less than that received for the cheaper seats. How many tickets of each kind were sold?

40. A man is paid at the rate of 5*s.* 3*d.* per hour for ordinary time, and 7*s.* 6*d.* per hour for overtime. He receives £14 18*s.* 6*d.* for a week in which he works a total of 53 hours. How many hours of ordinary time and of overtime did he work?

41. An article is produced at a cost of 9*s*., which includes the cost of the material and the cost of the labour involved. When the cost of the material rises by one-third and that of labour rises by one-quarter the cost of production increases by 2*s*. 7*d*. Find the original cost of the material and of the labour.

42. A manufacturer charges a fixed price per ton for his goods and delivers to his customers at a fixed charge per mile for every ton supplied. The total cost to a customer 10 miles from the factory for a load of 12 tons is £69, and the total cost to another customer 25 miles from the factory for a load of 8 tons is £49. Find the cost of the goods per ton and the cost of delivery per mile for each ton of goods.

---

43. A traveller does a journey of 140 miles in 6 hr., part of the way at 30 m.p.h., and the remainder at 20 m.p.h.

How far did he travel at each speed?

44. A man motored from A to B. Had he increased his speed by 10 m.p.h. he would have taken $1\frac{1}{4}$ hr. less; had he decreased his speed by 5 m.p.h. he would have taken 1 hr. more. Find how long he took for the journey and his speed in miles per hour.

45. A man can row a distance of 3 miles downstream in 45 min. and takes 1 hr. 30 min. to return. Find the speed of the current and the man's rowing speed in still water.

46. A man walked a certain distance. Had he walked 1 m.p.h. faster he would have taken 50 min. less, and had he walked $\frac{1}{2}$ m.p.h. slower he would have taken 40 min. longer. Find the distance he walked and his rate of walking.

(Let x hr. = time for journey and y m.p.h. = his speed.)

---

47. The perimeter of a rectangle is 28 ft. If its length is increased by 40% and its breadth by 50% the perimeter is increased by 12 ft. Find the length and breadth of the original rectangle.

48. A sum of money is shared between A and B. When A gives B £18, B has four times as much money as A. If B had given A £12 the two shares would have been equal. How much money has each?

49. A man receives a dividend of 9*d*. in the £ on his investment, and his wife receives a dividend of 1*s*. in the £. If their total investment is £660 and their total dividend is £28 10*s*., find the amount of each investment.

50. In a game of chance a boy pays 6*d*. for 4 tries. In addition, he receives 3*d*. for each success and pays an additional penny for each failure. He finds that his total loss is 2*s*. If, however, he had reversed his successes and his failures he would neither have gained nor lost. How many tries did he have and how many of them were successful?

Chapter 17

MULTIPLICATION AND DIVISION BY BINOMIALS AND TRINOMIALS

Multiplication by Binomials (i.e. expressions with *two* terms)

Consider $(x + a)(y + b)$.

$$\begin{aligned}(x + a)(y + b) &= (x + a)y + (x + a)b \\ &= xy + ay + xb + ab\end{aligned}$$

This result can be illustrated as follows:

x''	a''	
xy sq. in.	ay sq. in.	y''
xb sq. in.	ab sq. in.	b''

The length of the large rectangle is $(x + a)$ in.
The breadth of the large rectangle is $(y + b)$ in.

$$\therefore \text{ its area} = (x + a)(y + b) \text{ sq. in.}$$

But the area of the large rectangle is equal to the sum of the areas of the four small rectangles, namely,

$$(xy + ay + xb + ab) \text{ sq. in.}$$
$$\therefore (x + a)(y + b) = xy + ay + xb + ab$$

Similarly,

$$\begin{aligned}(x + a)(y - b) &= (x + a)y - (x + a)b \\ &= xy + ay - xb - ab\end{aligned}$$

and $\quad (x - a)(y + b) = xy - ay + bx - ab$

and $\quad (x - a)(y - b) = xy - ay - bx + ab$

Example 1: Multiply $(3x + 2)$ by $(2x - 3)$.

$$\begin{aligned}(3x + 2)(2x - 3) &= 2x(3x + 2) - 3(3x + 2)\\ &= 6x^2 + 4x - 9x - 6\\ &= 6x^2 - 5x - 6\end{aligned}$$

Note: After a little practice the answer will be written down at *once.*

Example 2: Multiply $3x^2 - 4x - 7$ by $2x - 3$.

Method I

$$\begin{aligned}&(3x^2 - 4x - 7)(2x - 3)\\ &\quad = 2x(3x^2 - 4x - 7) - 3(3x^2 - 4x - 7)\\ &\quad = 6x^3 - 8x^2 - 14x - 9x^2 + 12x + 21\\ &\quad = 6x^3 - 17x^2 - 2x + 21\end{aligned}$$

Method II

$$\begin{array}{rrrrl} & 3x^2 & -4x & -7 & \\ & & 2x & -3 & \\ \hline 6x^3 & -8x^2 & -14x & & \text{[the product of } 3x^2 - 4x - 7 \text{ and } 2x] \\ & -9x^2 & +12x & +21 & \text{[the product of } 3x^2 - 4x - 7 \text{ and } -3] \\ \hline 6x^3 & -17x^2 & -2x & +21 & \text{[by addition]} \end{array}$$

To check that the answer is correct we can substitute any value for x in the multiplicand, the multiplier and the product, as shown below.

When $x = 1$,

$$\begin{aligned}3x^2 - 4x - 7 &= 3(1)^2 - 4(1) - 7 = 3 - 4 - 7 = -8\\ 2x - 3 &= 2(1) - 3 = -1\\ \therefore\ (3x^2 - 4x - 7)(2x - 3) &= -8 \times -1 = +8\\ 6x^3 - 17x^2 - 2x + 21 &= 6(1)^3 - 17(1)^2 - 2(1) + 21\\ &= 6 - 17 - 2 + 21\\ &= +8\end{aligned}$$

$\therefore$ Answer is correct.

Note: It will be found convenient to arrange the terms in ascending order or in descending order before multiplying, e.g. $(2x - 3x^3 + 1 - 4x^2)(3 - 2x)$ should be arranged thus:

either $(1 + 2x - 4x^2 - 3x^3)(3 - 2x)$
or $(-3x^3 - 4x^2 + 2x + 1)(-2x + 3)$

Exercises 45

Find the following products:

1. $(x + 2)(x + 3)$
2. $(x + 3)(x + 4)$
3. $(y + 8)(y + 6)$
4. $(y + 4)(y - 2)$
5. $(a + 5)(a - 7)$
6. $(a - 6)(a + 3)$
7. $(x - 9)(x + 10)$
8. $(x - 5)(x - 4)$
9. $(x - 7)(x - 1)$
10. $(t - 1)(t - 1)$
11. $(t - 3)(t - 3)$
12. $(a + 2)(a - 2)$
13. $(a - 5)(a + 5)$
14. $(2 + x)(3 + x)$
15. $(6 + x)(4 + x)$
16. $(8 - x)(2 + x)$
17. $(5 + x)(3 - x)$
18. $(6 - x)(5 - x)$
19. $(4 + x)(4 - x)$
20. $(2 - x)(2 - x)$
21. $(x + 2y)(x + y)$
22. $(x + 4y)(x + 3y)$
23. $(a + 5b)(a - 2b)$
24. $(a - 7b)(a + 3b)$
25. $(a - 3b)(a + 3b)$
26. $(x - 2y)(x - 2y)$
27. $(y + 2x)(y - 3x)$
28. $(2x + 1)(x + 2)$
29. $(3x + 2)(2x + 3)$
30. $(2x + 1)(2x + 1)$
31. $(4x - 3)(4x + 3)$
32. $(3x - 4)(2x + 3)$
33. $(5x + 3)(2x - 5)$
34. $(2x - 3)(3x - 4)$
35. $(5x - 2)(4x - 3)$
36. $(2 + 3x)(1 - 2x)$
37. $(6 - 5x)(3 + 2x)$
38. $(2 - 5x)(2 - 5x)$
39. $(3 - 4x)(3 + 4x)$
40. $(3x + y)(x + 2y)$
41. $(4a + 3b)(2a + 5b)$
42. $(5a - 2b)(3a + 4b)$
43. $(7s + t)(4s - 3t)$
44. $(3s + 2t)(3s + 2t)$
45. $(3x - 4y)(3x + 4y)$
46. $(7x - 2y)(7x - 2y)$
47. $(x^2 + 3)(x^2 + 2)$
48. $(x^2 - 5)(x^2 + 4)$
49. $(ab - 2)(ab + 7)$
50. $(3 - x^2)(9 + x^2)$
51. $(2x^2 + y^2)(3x^2 - y^2)$
52. $(3x^2 + 2y^2)(2x^2 + 3y^2)$
53. $(a^3 - 5)(a^3 + 2)$
54. $(3a - b^3)(2a + b^3)$
55. $(9 + 2ab)(5 - 3ab)$
56. $(x^2 + 3x + 1)(x + 2)$
57. $(x^2 - 4x + 3)(x + 4)$
58. $(x^2 - 5x - 2)(x + 3)$

59. $(a^2 + a - 4)(a - 5)$
60. $(a^2 - a + 1)(a + 1)$
61. $(a^2 + a + 1)(a - 1)$
62. $(3x^2 + 2x + 1)(x + 2)$
63. $(2x^2 - 3x + 5)(2x + 3)$
64. $(a^2b^2 - 3ab - 2)(ab - 3)$
65. $(2x^2y^2 + 5xy - 1)(2xy + 3)$
66. $(5t^2 - 3t + 4)(2t - 5)$
67. $(3 - 2t + 2t^2)(3t - 2)$
68. $(7 + 4a^2 - 3a)(5a - 3)$
69. $(2x^2 - 3xy + 2y^2)(3x + 2y)$
70. $(2x^3 + x^2 - 4x - 1)(x - 3)$
71. $(3x^3 - 2x + 4x^2 - 2)(x + 4)$
72. $(2x - 5x^3 + 3)(2x - 3)$
73. $(9a^2 - 6ab + 4b^2)(3a + 2b)$

74. Multiply $3x^2 - 2x + 4$ by $2x - 3$, and check the result when $x = 2$.

75. Multiply $2 - x + 3x^3 - 2x^2$ by $3x - 4$, and check the result when $x = 1$.

76. Multiply $2a^2 - 3ab + 4b^2$ by $2a - b$, and check the result when $a = 1$, $b = -2$.

77. Multiply $4ab + 3b^2 - 4a^2$ by $3a + 5b$, and check the result when $a = 2$, $b = 1$.

78. Multiply $3 - 2x^3 + 2x$ by $2 - 3x$, and check the result when $x = -2$.

79. Find the coefficient of x^2 in the products:

(i) $(x + 2)(x^2 - 3x + 1)$
(ii) $(x - 4)(2x^2 + 4x - 3)$
(iii) $(3 - 2x)(5 - 3x + 2x^2 - x^3)$

80. Find the coefficient of x in the products:

(i) $(x + 3)(x^2 + 2x + 5)$
(ii) $(2x - 3)(x^2 - 5x - 2)$
(iii) $(5 + 4x)(x^3 - 2x^2 + 3x - 4)$

81. If $(3x^2 - 7x + 7)(x - 1) = 3x^3 - 10x^2 + ax - 7$, find a.

82. If $(5 + 5x^2 - 2x)(2x - 3) = 16x - 15 + ax^2 + 10x^3$, find a.

83. If the coefficient of x in the product of $x^2 + 5x + a$ and $x + 2$ is zero, find a.

84. If the coefficient of x^2 in the product of $(2x^2 - 3x - 2)$ and $(ax + 4)$ is zero, find a.

Special Cases

$$(\boldsymbol{a + b})^2 = (\boldsymbol{a + b})(\boldsymbol{a + b}) = \boldsymbol{a}^2 + 2\boldsymbol{ab} + \boldsymbol{b}^2$$
$$(\boldsymbol{a - b})^2 = (\boldsymbol{a - b})(\boldsymbol{a - b}) = \boldsymbol{a}^2 - 2\boldsymbol{ab} + \boldsymbol{b}^2$$
$$(\boldsymbol{a + b})(\boldsymbol{a - b}) = \boldsymbol{a}^2 - \boldsymbol{b}^2$$

The above results should be committed to memory, and may be expressed in words as follows:

1. The square of the sum of two numbers is equal to the sum of their squares together with twice their product.

2. The square of the difference of two numbers is equal to the sum of their squares diminished by twice their product.

3. The product of the sum and difference of two numbers is equal to the difference of their squares.

The above results are often written thus:

$$(a + b)^2 \equiv a^2 + 2ab + b^2$$
$$(a - b)^2 \equiv a^2 - 2ab + b^2$$
$$(a + b)(a - b) \equiv a^2 - b^2$$

The sign $\equiv$ means "is identically equal to", and is used when two expressions are equal for all values of the letters. Such an equation is called an *identity*, as opposed to an equation such as $3a = 6$, which is true for only *one* value of a. An equation such as $a + b = 4$ is true for various sets of values of a and b, but it is not true for all values of a and b. Equations such as $3a = 6$ and $a + b = 4$ are called *equations of condition* or briefly *equations*. $a + b = 4$ is an *indeterminate* equation, since there is an unlimited number of values of a and b for which it is true.

Exercises 46

Write down answers to the following:

1. $(x + 2)^2$
2. $(x + 5)^2$
3. $(a + 4)^2$
4. $(3 + a)^2$
5. $(10 + t)^2$
6. $(2x + 3)^2$
7. $(3x + 2)^2$
8. $(1 + 4x)^2$
9. $(2 + 3a)^2$
10. $(2x + y)^2$
11. $(3t + 4s)^2$
12. $(5a + 2b)^2$
13. $(ab + 2)^2$
14. $(x^2 + 3y)^2$
15. $\left(x + \frac{1}{x}\right)^2$
16. $\left(a^2 + \frac{1}{a^2}\right)^2$
17. $(2ab + 3c)^2$
18. $(x - 3)^2$
19. $(x - 7)^2$
20. $(a - 10)^2$
21. $(5 - a)^2$
22. $(12 - t)^2$
23. $(2x - 5)^2$
24. $(3s - 7)^2$
25. $(7t - 5)^2$
26. $(2 - 3a)^2$
27. $(5 - 6x)^2$
28. $(2x - y)^2$
29. $(4x - 3y)^2$
30. $(ab - 2c)^2$
31. $\left(x - \frac{1}{x}\right)^2$
32. $\left(a^2 - \frac{1}{a}\right)^2$
33. $(x^2 - 2x)^2$
34. $(2a^2 - 3a)^2$
35. $(5x + 2y)^2$
36. $\left(1 - \frac{1}{x}\right)^2$
37. $(1 + 2pq)^2$
38. $3(2x - 7y)^2$
39. $[2s(s - a)]^2$

Simplify:

40. $(20 + 1)^2$
41. $(40 - 1)^2$
42. $(100 + 2)^2$
43. $(100 - 3)^2$
44. $(10 + 0{\cdot}2)^2$
45. $(10 - 0{\cdot}01)^2$

Without actual multiplication find the value of:

46. 103^2
47. 98^2
48. $(9{\cdot}9)^2$
49. $(10{\cdot}4)^2$
50. $(9{\cdot}98)^2$
51. $(29{\cdot}9)^2$

What terms must be added to the following expressions to make each a perfect square?

52. $a^2 + 4a$
53. $x^2 + 6x$
54. $a^2 - 8a$
55. $x^2 - 10x$
56. $a^2 + 3a$
57. $x^2 - 5x$
58. $a^2 + 1$
59. $x^2 + 25$
60. $a^2 + 6\frac{1}{4}$
61. $4x^2 + 12x$
62. $9a^2 - 24a$
63. $25x^2 + 4$

Simplify:

64. $(x+1)(x-1)$
65. $(x+3)(x-3)$
66. $(x-2)(x+2)$
67. $(4+a)(4-a)$
68. $(5-a)(5+a)$
69. $(x+a)(x-a)$
70. $(x-y)(x+y)$
71. $(xy-3)(xy+3)$
72. $(2x-1)(2x+1)$
73. $(x^2-3)(x^2+3)$
74. $(5x+2)(5x-2)$
75. $(2ab-3c)(2ab+3c)$
76. $\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$
77. $(x^2-2x)(x^2+2x)$
78. $(ab-3c^2)(ab+3c^2)$

Simplify:

79. $(100+1)(100-1)$
80. $(10+0{\cdot}2)(10-0{\cdot}2)$
81. $(10+0{\cdot}03)(10-0{\cdot}03)$

Without actual multiplication find the value of:

82. 102×98
83. $10{\cdot}3 \times 9{\cdot}7$
84. $20{\cdot}02 \times 19{\cdot}98$
85. 63×57

Simplify:

86. $(x+3)(x+2)-(x-4)(x-1)-5(2x-1)$
87. $(x+5)^2-(x-2)(x+2)-(x-3)^2$
88. $(3x+2)^2-(2x+5)(x+2)+(2x-5)^2$
89. $x(2x-1)^2+2(3-4x)(3+4x)-2(x-2)(2x^2-3x+3)$
90. $(a+3b)^2-(5a+2b)(a-3b)+(2a-b)^2$

Prove:

91. $(a^2+a+1)(a-1)-(a^2-a+1)(a+1)=-2$
92. $(x+y)^2-(x-y)^2=4xy$
93. $(a+b)(a-b)+(b+c)(b-c)+(c+a)(c-a)=0$
94. $(a-b)^2+(b-c)^2-(c-a)^2=2(b-a)(b-c)$

Solve:

95. $(x+3)(x+1)=(x+2)(x+5)$
96. $(3x+2)(2x-7)-2(3x-5)(x-1)=0$
97. $(4x-3)^2-(x-2)^2-4(2x+5)^2+x^2=5$

98. $(5x + 7)(5x - 7) - (3x - 11)(3x + 11)$
$$= (4x - 9)(4x + 9) + 7x$$

99. $(3x^2 + 7x - 2)(2x - 3) = (x^2 + 3x - 2)(6x - 13) - 7$

100. $a(x - 1)^2 + b(x - 2)^2 = (a + b)x^2 - a$

Division by Binomials

Example: Divide $6x^3 - x^2 - 22x + 15$ by $2x - 3$.

The working can be compared with a division problem in Arithmetic.

$$\begin{array}{rl}
 & 345 \\
23) & 7935 \\
 & 69 \quad 23 \times 3 \\
\hline
 & 103 \\
 & 92 \quad 23 \times 4 \\
\hline
 & 115 \\
 & 115 \quad 23 \times 5 \\
\hline
\end{array}
\qquad
\begin{array}{rll}
 & 3x^2 + 4x - 5 & \\
2x - 3) & 6x^3 - x^2 - 22x + 15 & \\
 & 6x^3 - 9x^2 & (2x - 3) \times 3x^2 \\
\hline
 & 8x^2 - 22x & \\
 & 8x^2 - 12x & (2x - 3) \times 4x \\
\hline
 & -10x + 15 & \\
 & -10x + 15 & (2x - 3) \times -5 \\
\hline
\end{array}$$

If this is correct, then

$$(3x^2 + 4x - 5) \times (2x - 3) = 6x^3 - x^2 - 22x + 15$$

for all values of x. Its correctness can be checked as before by choosing any value for x and finding the value of each side of this equation.

When $x = -2$,

$$3x^2 + 4x - 5 = 3(-2)^2 + 4(-2) - 5 = 12 - 8 - 5 = -1$$
$$2x - 3 = 2(-2) - 3 = -4 - 3 = -7$$
$$\therefore\ (3x^2 + 4x - 5)(2x - 3) = (-1) \times (-7) = +7$$
$$\begin{aligned} 6x^3 - x^2 - 22x + 15 & \\ &= 6(-2)^3 - (-2)^2 - 22(-2) + 15 \\ &= -48 - 4 + 44 + 15 \\ &= 7 \end{aligned}$$

$\therefore$ Answer is correct.

Exercises 47

Divide:

1. $x^2 + 9x - 22$ by $x - 2$
2. $a^2 - 5a - 36$ by $a + 4$
3. $14 - 5x - x^2$ by $7 + x$
4. $6a^2 + 11a - 10$ by $3a - 2$
5. $15a^2 - 22a + 8$ by $5a - 4$
6. $15x^2 + x - 40$ by $3x + 5$
7. $25x^2 - 64$ by $5x + 8$
8. $14 - 5a - 24a^2$ by $2 - 3a$
9. $9 - 38a + 8a^2$ by $1 - 4a$
10. $10x^2 + 3xy - 18y^2$ by $2x + 3y$
11. $20x^2 - 51xy + 28y^2$ by $5x - 4y$
12. $x^3 + 5x^2 + 8x + 4$ by $x + 2$
13. $x^3 - 5x^2 + 3x + 9$ by $x - 3$
14. $10 - 8a^2 + 3a^3 - 37a$ by $a - 5$
15. $6a^3 - 7a^2 - 29a - 12$ by $3a + 4$
16. $16 - 8a - 19a^2 + 5a^3$ by $4 - 5a$
17. $8x^3 - 1$ by $2x - 1$
18. $9x^3 - 16x + 15$ by $3x + 5$
19. $10x^3 - 27x^2 + 27$ by $2x - 3$
20. $6x^2y^2 - xy - 12$ by $3xy + 4$

Find the quotient and remainder on dividing:

21. $15a^2 + a - 14$ by $5a + 2$
22. $12x^2 - 29x + 20$ by $4x - 3$
23. $35 + 13x - 10x^2$ by $7 - 3x$
24. $8a^3 - 6a^2 + 3a$ by $2a - 5$
25. $6a^3 - 2a^2 + 7a - 12$ by $3a - 4$
26. $20x^3 - 3x^2 - 19x - 16$ by $5x + 3$
27. $12 + x^2 - 5x - 6x^3$ by $3 - 2x$
28. Divide $4x^2 + 13xy + 3y^2$ by $4x + y$
29. Divide $8a^3 + 6a^2b + 3ab^2 + b^3$ by $2a + b$. Check the answer when $a = 1$, $b = 2$.

30. Divide $3x^3 - 8x^2y + xy^2 + 2y^3$ by $3x - 2y$. Check the answer when $x = y = 1$.

31. Divide $2a^2b^2 + abc - 3c^2$ by $ab - c$. Check the answer when $a = 1$, $b = 2$, $c = 3$.

32. Divide the sum of $2x(x + 1)$ and $2(x - 3)$ by $(x - 1)$.

33. Simplify $\dfrac{(3x + 2)(2x - 3)}{(2x + 5)}$.

34. What is the value of a if $2x^3 + x^2 + 3x + a$ is exactly divisible by $x + 2$.

35. What is the value of k if $3x^3 + 2x^2 - 13x + k$ is exactly divisible by $3x - 1$.

36. What must be added to $6a^2 - 13a$ to make it exactly divisible by $2a - 3$?

37. The expression $x^3 - 6x^2 + 11x + a$ is exactly divisible by $x - 2$. Find a.

38. If $a = (x - 1)^2$, $b = (x - 1)(x + 1)$, $c = x - 1$, simplify $\dfrac{3a + 2b}{5c + 4}$.

39. If $a = (x - 2)^2$, $b = (x - 2)(x + 2)$, $c = 3(x - 2)$, simplify $\dfrac{2a - b}{c - 2x}$.

40. If $a = (x - 1)(x - 2)(x - 3)$, $b = (x - 1)(x - 2)$, $c = x - 1$, simplify $\dfrac{a + b + c}{b - c + 2}$

Multiplication and Division by Expressions with Three or More Terms

Where expressions with three or more terms are concerned the methods are the same as those used with binomials. In general, care should by taken:

(1) to arrange the expressions in ascending or descending order (where more than one letter is involved the order is taken for any one letter);

(2) to place like terms underneath one another;

(3) to avoid crowding the work.

Multiplication

Example 1: Multiply $a^3 + 5 - 2a^2$ by $3a - 4 + 5a^2$.

$$\begin{array}{rrrrrr}
 & & a^3 & -2a^2 & & +5 \\
 & & & +5a^2 & +3a & -4 \\
\hline
5a^5 & -10a^4 & & +25a^2 & & \\
 & +3a^4 & -6a^3 & & +15a & \\
 & & -4a^3 & +8a^2 & & -20 \\
\hline
5a^5 & -7a^4 & -10a^3 & +33a^2 & +15a & -20 \\
\hline
\end{array}$$

[A space is left because there is no term in a.]

Example 2: Expand $(a + b - c)^2$.

$$\begin{array}{rrrrrr}
a & +b & -c & & & \\
a & +b & -c & & & \\
\hline
a^2 & +ab & -ac & & & \\
 & +ab & & +b^2 & -bc & \\
 & & -ac & & -bc & +c^2 \\
\hline
a^2 & +2ab & -2ac & +b^2 & -2bc & +c^2 \\
\hline
\end{array}$$

Example 3: In the product $(2x^2 + 3x + a)(x^2 - 2x - 4)$ the coefficient of x^2 is -9. Find a. Then find the coefficient of x.

$$(2x^2 + 3x + a)(x^2 - 2x - 4)$$

The partial products involving x^2 come from the multiplication of the terms linked by the continuous lines shown above, namely, $+a \times x^2$, $+3x \times -2x$, $2x^2 \times -4$

$$\begin{aligned}
\therefore \text{ coefficient of } x^2 &= +a - 6 - 8 \\
&= +a - 14 \\
\text{But coefficient of } x^2 &= -9 \\
\therefore\ +a - 14 &= -9 \\
\therefore\ +a &= -9 + 14 \\
\therefore\ a &= 5
\end{aligned}$$

The partial products involving x come from the multiplication of the terms linked by the dotted lines shown above, namely, $+a \times -2x$, $+3x \times -4$.

$$\therefore \text{ coefficient of } x = -2a - 12$$
$$\text{But } a = 5$$
$$\therefore \text{ coefficient of } x = -10 - 12$$
$$= -22$$

Exercises 48

Multiply:

1. $x^2 - 2x + 3$ by $x^2 + 3x - 2$, and check the result when $x = 1$

2. $2a^2 - a - 3$ by $a^2 - 2a + 1$, and check the result when $a = 2$.

3. $3y^2 + 2y - 5$ by $2y^2 - 4y - 3$, and check the result when $y = -2$.

4. $5a^2 - 3a + 2$ by $a^2 - 2a + 4$, and check the result when $a = -1$.

5. $3 - 2x + x^2$ by $5 + 3x - 2x^2$

6. $4 + x - 3x^2$ by $2 - x + 3x^2$

7. $x^3 + 2x - 3$ by $x^2 - x + 2$

8. $2y^3 - 5 + 3y^2$ by $2 - 3y^2 + 4y$

9. $a^2 - ab + 2b^2$ by $a^2 + 2ab - 3b^2$

10. $3a^2 + 5ab - 4b^2$ by $2a^2 - 3ab - b^2$

11. $a^2 + b^2 - ab$ by $a^2 + b^2 + ab$

12. $3b^2 - 2ab + 3a^2$ by $ab - 2a^2 + 3b^2$

13. $x - y - z$ by $x + y - z$

14. $2a + 3b - c$ by $3a - 2b + c$

15. $2b - 5c - a$ by $3c + b - 4a$

16. $x^2 + y^2 + z^2 - xy - yz - zx$ by $x + y + z$

17. $a^2 - \frac{1}{a} + 1$ by $2 + \frac{1}{a} - a$

18. $2a^2 - 1 + \frac{3}{a}$ by $\frac{2}{a} - 3 + a$

Find:

19. The coefficient of x in the product
$$(3x^2 - 2x + 4)(2x^2 - 3x - 2).$$

20. The coefficient of x^3 in the product
$$(2x^3 - 3x^2 + 4x - 5)(3x^2 - 7x + 4).$$

21. The coefficient of xy in the product
$$(2x + 3y - 4)(3x - 2y + 5).$$

22. The coefficient of x^2 in $(3x^2 - 2x + 5)^2$.

23. The coefficient of a^2b^2 in $(2a^2 - 3ab - b^2)^2$.

24. The coefficient of y in the product
$$(ax + by + c)(bx - ay - c).$$

25. If the coefficient of x in the product
$(3x^2 - 4x + 5)(2x^2 - 3x + a)$ is -7, find a.

26. If the coefficient of x^2 in the product
$(7 + 2x + 4x^2)(3 + ax - 3x^2)$ is -19, find a.

27. If the coefficient of y in the product
$(3x - ay + 4)(ax + 3y - 2)$ is 16, find a.
Then find the coefficient of xy.

28. If the coefficient of x in $(5x^2 + ax - 2)^2$ is -12, find a.

29. If the coefficient of x^3y in $(3x^2 - axy + 3y^2)^2$ is 18, find a.

Simplify:

30. $(a + b)(b + c)(c + a)$

31. $(x + 2)(2x - 3)(3x + 1)$

32. $(3a^2 - 2a + 1)(a^2 - a - 1) - (a^2 + 2a - 1)^2$

33. $(a - 2b + c)(a + 2b - c)$
$$+ (2a - b - c)(2a + b + c) - 5(a^2 - b^2)$$

Products Expanded by the Use of Standard Forms

By suitable grouping an expansion can often be made to depend on standard forms already known.

Example 1:

$$(a+b+c)^2 = (a+b+c)^2$$
$$= (a+b)^2 + 2(a+b)c + c^2$$

[from the standard form $(x+y)^2 =$ etc.]

$$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2$$

usually written in the form

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Hence **the square of the sum of three (or more) numbers is equal to the sum of their squares together with twice the product of every pair.**

This result can be used in expansions like the following:

$$(2x - 3y - 4z)^2 = (2x)^2 + (-3y)^2 + (-4z)^2 + 2(2x)(-3y)$$
$$+ 2(-3y)(-4z) + 2(-4z)(2x)$$
$$= 4x^2 + 9y^2 + 16z^2 - 12xy + 24yz - 16zx$$

Example 2:

$$(x+y-z)(x-y+z) = (x+\overline{y-z})(x-\overline{y-z})$$
$$= x^2 - (y-z)^2$$

[from the standard form $(a+b)(a-b) = a^2 - b^2$]

$$= x^2 - y^2 - z^2 + 2yz$$

Example 3: $(2x-3y-5)(3x+y+2)$

$$= (\overline{2x-3y}-5)(3x+y+2)$$
$$= (2x-3y)(3x+y) - 5(3x+y) + 2(2x-3y) - 10$$
$$= 6x^2 - 7xy - 3y^2 - 15x - 5y + 4x - 6y - 10$$
$$= 6x^2 - 7xy - 3y^2 - 11x - 11y - 10$$

Example 4:

$$(a+b)^3 = (a+b)^2(a+b)$$
$$= (a^2 + 2ab + b^2)(a+b)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

Similarly, $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.
Both these results should be memorised.

Hence $(2x + 3y)^3$

$$= (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3$$

[putting $2x$ for a and $3y$ for b in $(a+b)^3$]

$$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$$

Example 5: $(x + a)(x + b)(x + c)$

$$= (x^2 + ax + bx + ab)(x + c)$$
$$= x^3 + ax^2 + bx^2 + abx + x^2c + acx + bcx + abc$$
$$= x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$$

Note that the coefficient of x^2 is the sum of a, b, c, and the coefficient of x is the sum of the products of a, b and c taken two at a time.

Hence $(x + 2)(x + 3)(x - 4)$

$$= x^3 + (2 + 3 - 4)x^2 + (6 - 12 - 8)x - 24$$
$$= x^3 + x^2 - 14x - 24$$

Exercises 49

Simplify:

1. $(a + b - c)^2$
2. $(a - b - c)^2$
3. $(2a + b + 3c)^2$
4. $(a - 2b - 3c)^2$
5. $(1 - x + 2y)^2$
6. $(2 - 3x - y)^2$
7. $(2a^2 + a - 1)^2$
8. $(x^2 - xy + 3y^2)^2$
9. $(a + b + c)(a - b - c)$
10. $(a + b + c)(a + b - c)$
11. $(x + 3y + 2)(x + 3y - 2)$
12. $(3a - 2b + c)(3a + 2b + c)$
13. $(1 - a - b)(1 - a + b)$
14. $(x + y + 2)(x + 2y - 2)$
15. $(x + 2y - 5)(x - 3y + 2)$
16. $(1 - x + x^2)(1 + x - x^2)$
17. $(x + 3y + 4)(x - 2y - 5)$
18. $(x + 2)^3$
19. $(x + 3)^3$
20. $(3x + 1)^3$
21. $(2x + 5y)^3$
22. $(x - 1)^3$
23. $(x - 4)^3$
24. $(2x - 3)^3$
25. $(3x - 2y)^3$
26. $(2 - 3a)^3$

27. $\left(a + \dfrac{1}{a}\right)^3$

28. $\left(x - \dfrac{1}{x}\right)^3$

29. $(x + 1)(x + 2)(x + 3)$

30. $(x + 3)(x + 2)(x + 4)$

31. $(x - 1)(x - 3)(x - 5)$

32. $(x + 2)(x - 3)(x + 1)$

33. $(x - 3)(x - 1)(x + 4)$

34. $(x - a)(x - b)(x - c)$

35. $(x - a)(x + b)(x - c)$

36. $\left(x + \dfrac{1}{x}\right)^3 + \left(x - \dfrac{1}{x}\right)^3$

37. $2(a - b - c)^2 + (b - c - a)^2 - (c - a - b)^2$

Division

Example 1: Divide $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$

$$
\begin{array}{r|l}
 & x^2 + xy + y^2 \\
\hline
x^2 - xy + y^2 & x^4 \qquad\qquad + x^2y^2 \qquad\qquad + y^4 \\
 & \underline{x^4 - x^3y + x^2y^2} \\
 & \quad + x^3y \\
 & \quad \underline{+ x^3y - x^2y^2 + xy^3} \\
 & \qquad\qquad + x^2y^2 - xy^3 + y^4 \\
 & \qquad\qquad \underline{+ x^2y^2 - xy^3 + y^4}
\end{array}
$$

[Spaces are left because there are no terms in x^3y or in xy^3.]

Example 2: Find the quotient and remainder when $x^4 - 3x^3 + 5x^2 - 6$ is divided by $x^2 - 2x + 2$.

$$
\begin{array}{r|l}
 & \qquad\qquad\qquad x^2 - x + 1 \\
\hline
x^2 - 2x + 2 & x^4 - 3x^3 + 5x^2 \qquad\quad - 6 \\
 & \underline{x^4 - 2x^3 + 2x^2} \\
 & \quad - x^3 + 3x^2 \\
 & \quad \underline{- x^3 + 2x^2 - 2x} \\
 & \qquad\qquad x^2 + 2x - 6 \\
 & \qquad\qquad \underline{x^2 - 2x + 2} \\
 & \qquad\qquad\qquad 4x - 8
\end{array}
$$

The division stops when the remainder obtained is of lower degree than the divisor.

$\therefore$ Quotient is $x^2 - x + 1$ and the remainder is $4x - 8$.

G

Exercises 50

Divide:

1. $2x^3 - x^2 + 3x + 2$ by $x^2 - x + 2$
2. $6a^3 - 19a^2 + a + 6$ by $2a^2 - 5a - 3$
3. $6k^3 - 34k + 20$ by $3k^2 + 6k - 5$
4. $2a^4 - 5a^3 - 3a^2 + 14a - 8$ by $a^2 - 3a + 2$
5. $14a + 9a^4 - a^2 - 12$ by $2a + 3a^2 - 3$
6. $2 + 6x^4 - 6x^2 - 7x^3 + 5x$ by $3x^2 - 2x - 1$
7. $x^4 + x^2 + 1$ by $x^2 + x + 1$
8. $a^4 + a^3b - 3a^2b^2 + 7ab^3 - 6b^4$ by $a^2 + 2ab - 3b^2$
9. $x^6 - 1$ by $x^2 - x + 1$
10. $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$
11. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$
12. $x^3 + y^3 - z^3 + 3xyz$ by $x + y - z$

Find the quotient and remainder when:

13. $x^4 - 3x^3 + 2x - 4$ is divided by $x^2 + x - 3$.
14. $2y^4 - 3y^2 + y - 5$ is divided by $2y^2 - 4y + 3$.
15. $6a^4 + a^3b - 6a^2b^2 + 7ab^3 - 6b^4$ is divided by $3a^2 - ab + 2b^2$.
16. $8x^3 + 6x^2 - 4$ is divided by $4x^2 - 3x - 2$.
17. If $3x^2 + 4x - 5$ is a factor of

$$6x^4 - 13x^3 - 29x^2 + 47x - 15$$

find the other factor

18. Divide the product of $3x^2 - 8x + 4$ and $2x^2 + 5x + 3$ by $2x^2 - x - 6$.
19. What expression divided by $2x^2 + 3x - 4$ will give a quotient of $x^2 - x - 2$ and a remainder of $2x - 5$?
20. Divide $8x^3 - y^3 - 1 - 6xy$ by $2x - y - 1$.

Paper 21

1. (*a*) Multiply $2x^2 - 3x + 5$ by $3x - 2$.

(*b*) Divide $3x^3 - 2x^2 - 25x + 12$ by $x - 3$.

2. Solve $2x + y = 3x - 2 = 4y + 1$.

3. The first of two numbers is less than the second by 5. Three times the smaller number exceeds twice the larger by 8. Find the numbers.

4. Simplify:

(i) $(a - 4)^2 + 3(a - 3)(a + 3) - (a + 2)^2$

(ii) $(2a + 3)(3a - 2) - (1 - 4a)(2 + a)$.

5. Solve $\dfrac{2x}{3} - \dfrac{5}{6} = 2\left(\dfrac{x}{4} - 3\right) - 3\left(\dfrac{x}{5} - 2\frac{1}{2}\right)$.

6. A boy takes x min. less to cycle to school than to walk. If he cycles at 10 m.p.h. and walks at 3 m.p.h., how far away is his school? How far away is the school when $x = 21$?

Paper 22

1. Solve
$$2x - \frac{3y}{5} = 2$$
$$2y + \frac{3x}{2} = 26$$

2. Simplify:

(i) $(2x^2 - 3xy + y^2) \times (2x - 3y)$

(ii) $(x^3 + 3x^2y - 3xy^2 - y^3) \div (x - y)$

3. If $p^2 = (3x - y)^2 - (x + 3y)(x - 3y) + (x + 3y)^2 - 2y(5y - 9x)$, express p, as simply as possible, in terms of x and y.

4. A man buys $(a + b)$ model aeroplanes at x shillings each and $(x - b)$ model aeroplanes at a shillings each. He sells all

of them at b shillings each. Find an expression for the Selling Price — the Cost Price in pounds.

Does the man always gain or lose? How do you know?

5. If $a = 1$, $b = \frac{1}{2}$, $c = -2$, $d = 0$, find the value of:

(i) $(a + b)(b + c)(c + d)$

(ii) $\dfrac{a^2 - b^2}{a - b} + \dfrac{b^2 - c^2}{b + c} + \dfrac{c^2 - d^2}{c - d}$

6. For a certain concert, tickets are sold at 5*s.* each and at 2*s.* 6*d.* each. Had those who bought 5*s.* tickets bought 2*s.* 6*d.* tickets instead, and vice versa, the receipts would have risen by £24 10*s.* How many people were at the concert, if the number who bought 2*s.* 6*d.* tickets was 340?

Paper 23

1. (i) Multiply $3x^2 - x + 2$ by $2x^2 + 3x - 1$.

(ii) Divide $10x^4 + 11x^3 - 17x^2 - 6$ by $5x^2 - 2x + 2$.

2. 25 boys and 15 girls sit an examination. The average mark of the boys is x and of the girls y. What is the average mark of all the candidates? It is decided to make the average mark 50, and to change each candidate's marks accordingly. By what fraction will the mark each gained in the examination have to be multiplied?

If $x = 48$ and $y = 40$, what would the new mark be of a girl who made 30 marks?

3. Solve
$$3x = 4y + 11$$
$$2x + 17y + 8{\cdot}4 = 0$$

4. (i) Find the coefficient of x^2 in the product
$$(2 - 3x)(3 - 5x + 2x^2 - x^3)$$

(ii) Find a, if the coefficient of x^2 in the product
$$(3x - 4)(2 + 3x + ax^2 - x^3)$$
is $+25$.

5. Find an equation of the form $ax + by + 1 = 0$, where

a and b are constants, and which is such that it has the two solutions $x = -2$, $y = 2$; $x = \frac{1}{2}$, $y = -5\frac{1}{2}$.

6. A committee of 18 has men and women members. If half of the women on the committee were replaced by men, the number of men would be five times the number of women. How many men are at present on the committee?

Paper 24

1. (i) Find the value of $2a^2 - \frac{b}{\ } + bc$, when $a = -\frac{1}{4}$, $b = \frac{1}{8}$, $c = -\frac{1}{3}$.

(ii) Find the value of

$$\left(\frac{x + 2y}{2}\right)^3 - \left(\frac{2x - y}{4}\right)^2$$

when $x = -2$, $y = 3$.

2. One side of a rectangle is $(2x - 3)$ in. long. Its area is $(6x^2 - x - 12)$ sq. in. Find its perimeter.

If its perimeter is 4 ft. 4 in., find its dimensions and its area.

3. A man sells in a week x articles at y shillings each. He reduces the price of each article by one-sixth and, as a result, the next week he increases the number of articles sold by one-sixth. Have his takings increased or decreased? By what fraction?

4. Solve:

(i) $$5(2t - 3)(2 + 3t) - (4t + 3)(4t - 3) = 14t(t - 3) + 13$$

(ii) $$\frac{5}{x} = \frac{3}{y} + 10, \quad \frac{2}{x} + 4\frac{1}{2} = 3 - \frac{1}{y}$$

5. (i) If the coefficient of x^2 is 16 in the product of $(2 - ax - 2x^2)$ and $(4 - 4x + 6x^2)$, find the value of a and then find the coefficient of x^3.

(ii) Prove that

$$(ax + by + c)^2 - (2ax + c)(2by + c) = (ax - by)^2$$

6. By increasing his usual speed by 10 m.p.h. a man reckons that he will save 25 min. on a certain journey. His car develops a fault, and as a result his average speed is decreased by 5 m.p.h. and he takes 20 min. longer than usual. How long does he usually take for the journey, and at what speed does he usually travel?

Paper 25

1. A man bought $(3x - y)$ toys at $(2x + y)$ shillings each. He sold $2x$ of them at $(3x - 2y)$ shillings each and the rest at $2x$ shillings each. Find an expression for his profit, and simplify it. If he bought 50 toys at £2 each, what was his profit?

2. The volume of a square prism is

$$12x^3 + 28x^2y + 3xy^2 - 18y^3$$

The length of the side of the square base is $(2x + 3y)$ in. Find the height of the prism.

3. Solve $$\frac{3x - 4}{5} - \frac{14y - 2}{6} = \frac{3x + 2y - 2}{3}$$

$$\frac{x + 2y}{3} = \frac{x - y}{4} + \frac{1}{3}$$

4. Simplify:

(i) $4a(2b - 3c) - 3b[2c - 4(a - 2b)]$

(ii) Find the value of a if $6x^3 - 17x^2 + 22x + a$ is exactly divisible by $2x - 3$.

5. One pump can empty a tank in t min., and a second pump in $2t$ min. If the tank contains 100 gal. when full, how long will it take to empty the full tank when both pumps are working?

6. For £7 a man can buy either 12 lb. tea and 6 lb. of coffee or 7 lb. tea and $10\frac{1}{2}$ lb. coffee. Find the price of 1 lb. tea and of 1 lb. coffee.

CHAPTER 18

FACTORS

I. *Common Factor*

Example 1: Factorise $ax + ab$.

This expression consists of two terms, ax and $+ab$

These terms have a common factor a.

$\therefore$ a is a factor of the expression $ax + ab$.

To find the other factor, $ax + ab$ is divided by a

$$\frac{ax + ab}{a} = x + b \quad \text{or} \quad \begin{array}{r} a)\overline{ax + ab} \\ x + b \end{array}$$

$\therefore$ The other factor is $x + b$.

$$\therefore\ ax + ab = a(x + b)$$

Note: In the expression $ax + ab$ there are *two* terms When this expression has been factorised there is *one* term. This term has *two factors*, namely a and $(x + b)$.

Example 2: Factorise $(a - b)^2 - (a - b)$.

This expression consists of two terms, $(a - b)^2$ and $-(a - b)$.

These terms have a common factor $(a - b)$.

$\therefore$ $(a - b)$ is a factor of the expression $(a - b)^2 - (a - b)$.

To find the other factor $(a - b)^2 - (a - b)$ is divided by $(a - b)$.

$$\frac{(a - b)^2 - (a - b)}{a - b} = (a - b) - 1 \quad \text{or} \quad \begin{array}{r} (a - b)\overline{(a - b)^2 - (a - b)} \\ (a - b) - 1 \end{array}$$

$\therefore$ The other factor is $(a - b) - 1$.

$$\therefore\ (a - b)^2 - (a - b) = (a - b)[(a - b) - 1]$$
$$= (a - b)(a - b - 1)$$

An expression is not factorised until it has been reduced to a single term.

E.g. $ax + ab + cx + cb$ can be written as

$$a(x + b) + c(x + b)$$

but it is not factorised because this expression has 2 terms.

$$\text{But } a(x + b) + c(x + b) = (x + b)(a + c)$$

The expression has now been factorised because there is *one* term only with *two* factors $(x + b)$ and $(a + c)$.

Exercises 51

Factorise:

1. $2x + 2y$
2. $3a - 3b$
3. $ax + ay$
4. $bx - by$
5. $x - bx$
6. $ak + a$
7. $-ab - bx$
8. $3a + 6$
9. $8 - 4x$
10. $abc - b$
11. $25 + 20x$
12. $ab + ac - ad$
13. $3a - 6b + 12c$
14. $2ax - 6ay$
15. $2axy + 8aby$
16. $a^2 + ab$
17. $2x^2 - x$
18. $x^3 + 3x^2 - 2x$
19. $6t^3 - 9t^2 + 12t$
20. $2a - 6a^2 + 10a^3$
21. $b^2a + bc^2$
22. $ax^2 - bx^4$
23. $-3k^4 + 6k^3 - 12k^2$
24. $a - \dfrac{a}{b}$
25. $a^2b + 2abc$
26. $2xy + 2xk - 4x^2$
27. $x^3 + 10x$
28. $x^2y^3 - x^3y^4$
29. $a^3 - 2a^2b + 3ab^2$
30. $\pi D - \pi d$
31. $2\pi r^2 + 2\pi rl$
32. $\frac{1}{3}x^2y - \frac{1}{3}xy^2$
33. $\frac{1}{2}a - 2a^2$
34. $24ab + 40a^2$
35. $121a^4 - 33a^2$
36. $-45t^5 + 9t^3$
37. $2a^3 - 3a^2 + 6a$
38. $x(a + 2) + y(a + 2)$
39. $2x(a - 3) - 5(a - 3)$
40. $k(a + c) - kb(a + c)$
41. $5a(x - y) + 3b(x - y)$
42. $r(s + l) + rs$
43. $r(s - t) - r^2(s - t)^2$
44. $x(x - 4) + (x - 4)$
45. $(x - 3)^2 - 4(x - 3)$

46. $3x(a - b + c) + y(a - b + c)$
47. $a(m + n) - b(m + n)^2$
48. $x(y - 1) - (y - 1)$
49. $a(b - c) - x(b - c) + y(b - c)$
50. $x^2(m - n) - 3x(m - n) + (m - n)$

II. *Grouping*

It is sometimes necessary to take the terms of an expression in groups before the expression can be factorised. The terms are grouped if they have a common factor.

Example 1: Factorise $ax - ay + bx - by$.

The first two terms are grouped because they have a common factor a.

The last two terms are grouped because they have a common factor b.

$$\begin{aligned} ax - ay + bx - by &= (ax - ay) + (bx - by) \\ &= a(x - y) + b(x - y) \end{aligned}$$

We have now an expression with two terms, and these two terms have a common factor $(x - y)$.

To find the other factor we divide $a(x - y) + b(x - y)$ by $(x - y)$. The result is $(a + b)$.

$$\therefore\ ax - ay + bx - by = (x - y)(a + b)$$

Example 2: Factorise $a^2 - b - a + ab$.

The first and third terms have a common factor a.

$$a^2 - a = a(a - 1)$$

The second and fourth terms have a common factor b, and when we divide them by b we must have $(a - 1)$ as the other factor. The a can come only from the 4th term, and so we must take the 4th term first. Hence

$$\begin{aligned} a^2 - b - a + ab &= (a^2 - a) + (ab - b) \\ &= a(a - 1) + b(a - 1) \\ &= (a - 1)(a + b) \end{aligned}$$

The example could, of course, have been worked as follows, grouping terms 1 and 3, and terms 2 and 4 in that order:

$$a^2 - b - a + ab = (a^2 - a) - (b - ab)$$

[*Note* the change of sign required.]

$$= a(a - 1) - b(1 - a)\ *$$
$$= a(a - 1) + b(a - 1)$$
$$= (a - 1)(a + b)$$

*$(1 - a)$ is *not* the same as $(a - 1)$. $(1 - a) = -(a - 1)$ and so we must rewrite the expression.

Careful consideration of the order of terms in grouping avoids this double change of sign.

Note I: Even if an expression can be written in groups of terms containing common factors, it may not be possible to factorise it.

E.g. $\quad ab - ac - cd + c^2 = a(b - c) - c(d - c)$

These two terms have no common factor, so the expression cannot be factorised.

Note II: $\qquad (a + b) = (b + a)$

Hence

$$x(a + b) + y(b + a) = (a + b)(x + y)$$

But $\qquad (a - b) = -(b - a)$

$x(a - b) + y(b - a)$ must be rewritten in the form

$$x(a - b) - y(a - b) = (a - b)(x - y)$$

Example 3: Factorise $4x - 4y + 2ay - 2ax$

$$\begin{aligned} 4x - 4y + 2ay - 2ax &= 2(2x - 2y + ay - ax) \\ &= 2[(2x - 2y) - (ax - ay)] \\ &= 2[2(x - y) - a(x - y)] \\ &= 2[(x - y)(2 - a)] \\ &= 2(x - y)(2 - a) \end{aligned}$$

Exercises 52

Factorise:

1. $a(x + y) + (bx + by)$
2. $ax - ay + bx - by$
3. $ax + bx + ay + by$
4. $2x + 6 + ax + 3a$
5. $ax - ab + 4x - 4b$
6. $3x + 3 + kx + k$
7. $st - 4t + 5s - 20$
8. $x^2 + 5x + ax + 5a$
9. $x^2 - ax + bx - ab$
10. $(ab + ac) - (bx + cx)$
11. $(rs + st) - kr - kt$
12. $x^2 + 2x - ax - 2a$
13. $a^2 - ab - ax + bx$
14. $5x - ax - 5y + ay$
15. $6x^2 - 4xy + 9x - 6y$
16. $a^3 + a^2 + a + 1$
17. $x^3 - x^2 + x - 1$
18. $2x^3 + 3x^2 + 2x + 3$
19. $x^2 + y + x + xy$
20. $a^2 - b + a - ab$
21. $pq + 1 + p + q$
22. $a^2 - 4ab + 8b - 2a$
23. $xy - 1 - y + x$
24. $2xz - 4yz + 6y - 3x$
25. $a^2 + 2ab - 2b - a$
26. $4xy - ab + 4ay - xb$
27. $a - a^2 - ab + b$
28. $2x^3 - x^2 + 8x - 4$
29. $3ac - bc + 3ab - 9a^2$
30. $a^2x + x^2a + x^3 + a^3$
31. $1 + 2x + 5x^2 + 10x^3$
32. $2 - 3a + 2a^2 - 3a^3$
33. $ab^2 - 2a^2 + 2b^3 - 4ab$
34. $b^3 - 3ab^2 - 2ab + 6a^2$
35. $(a + b)^2 - ac - bc$
36. $(x + y)^2 - x - y$
37. $pmn - qm^2 - p^2n + pqm$
38. $(a + b)(x + y) - 3a - 3b$
39. $5x^3 + 20x^2 - 7x - 28$
40. $x^2 - yz - xz + xy$
41. $3x - 3y + 6ay - 6ax$
42. $2ab + 3bc - 6b^2 - ac$
43. $abc + 3akc - bd - 3kd$
44. $2xy - 12 - 6x + 4y$
45. $2adx + abd - 2bz - 4xz$
46. $ax + ay + bx + by + cx + cy$
47. $x^3 - x^2 + xy + x - y - 1$
48. $1 - xy(1 - xy) - x^3y^3$
49. $x(3 - y) + x^2 - 3y$
50. $x(a - 3b) + y(a - x) - a(x - 3b)$

III. *Trinomials* (i.e. algebraic expressions with 3 terms)

A. Type $x^2 + ax + b$, where a and b are constants.

In Chapter 17 we saw that

$$(x + 6)(x + 2) = x^2 + 8x + 12$$

The first term x^2 is got by multiplying x by x.

The last term $+12$ is got by multiplying $+6$ by $+2$.

The middle term, i.e. the term in x, is got by multiplying the terms in the brackets connected by the links shown above, i.e. $+6$ and x, giving $+6x$, and x and $+2$, giving $+2x$, and adding these two results to get $+8x$.

Similarly,

$$(x + 6)(x - 2) = x^2 + 4x - 12$$

(the middle term is $6x - 2x = +4x$)

$$(x - 6)(x + 2) = x^2 - 4x - 12$$

(the middle term is $-6x + 2x = -4x$)

$$(x - 6)(x - 2) = x^2 - 8x + 12$$

(the middle term is $-6x - 2x = -8x$)

From these results we note:

(i) That the constant term is the algebraic product of the numerical terms inside the brackets.

(ii) That if the sign of the constant term is $+$, the signs inside the bracket are like.

(iii) That if the sign of the constant term is $-$, the signs inside the brackets are unlike.

(iv) That the coefficient of x is the algebraic sum of the numerical terms inside the brackets.

Hence the problem of factorising an expression of this type is reduced to the problem of finding two *numbers* satisfying the conditions stated. It has nothing to do with the x or y or z, etc., of the expression.

The expressions $x^2 + 9x + 18$, $y^2 + 9y + 18$, $z^2 + 9z + 18$ present exactly the same problem as far as factors are concerned:

Example 1: Factorise $x^2 + 9x + 18$.

Since the constant term is $+18$ and the coefficient of x is $+9$, we are looking for two numbers whose product is $+18$ and whose sum is $+9$, namely $+6$ and $+3$.

$$\therefore\ x^2 + 9x + 18 = (x + 6)(x + 3)$$

Obviously $\quad y^2 + 9y + 18 = (y + 6)(y + 3)$

Example 2: Factorise $x^2 - 7x + 12$.

We look for two numbers, whose product is $+12$ and whose sum is -7, namely -4 and -3.

$$\therefore\ x^2 - 7x + 12 = (x - 4)(x - 3)$$

Example 3: Factorise $3x^2 - 12x - 63$.

There is a common factor, 3.

$$\therefore\ 3x^2 - 12x - 63 = 3(x^2 - 4x - 21)$$

Considering the algebraic expression inside the brackets, we look for two numbers whose product is -21 and whose sum is -4, namely -7 and $+3$.

$$\therefore\ 3x^2 - 12x - 63 = 3(x - 7)(x + 3)$$

Note: 1. With practice it will be found that the factors can often be written down at once.

2. Factors of trinomials are often found in some such way as follows:

Example 4: Factorise $x^2 - 3x - 28$.

We can write down $(x \quad)(x \quad)$ as the form of the factors since the first term is x^2. Since the sign of the constant term is $-$, we know that the signs inside the brackets are unlike. Hence we can write $(x + \ ?\ )(x - \ ?\ )$. The numbers to be inserted must be factors of 28, and must therefore be 28 and 1 or 2 and 14 or 4 and 7. By trial and error we reject those factors

that will not give the middle term $-3x$, e.g. if we write $(x + 2)(x - 14)$ as the factors, the middle term would be $+2x - 14x = -12x$, and we see that these factors are wrong.

It will be found that $(x + 4)(x - 7)$ are the factors, since the middle term is $+4x - 7x = -3x$. This method is most useful in an example like Example 5 below.

Example 5: Factorise $(15 + 2x - x^2)$.

$$15 + 2x - x^2 = (? + x)(? - x)$$

The numbers to be inserted are factors of 15, and when inserted must give a middle term $+2x$.

$$\therefore\ 15 + 2x - x^2 = (3 + x)(5 - x)$$

[Middle term is $-3x + 5x = +2x$.]

This example should be factorised as shown above. The terms should not be rearranged in descending order.

Exercises 53

Factorise:

1. $x^2 + 5x + 6$
2. $x^2 + 7x + 12$
3. $a^2 + 8a + 15$
4. $a^2 + 11a + 28$
5. $x^2 + 6x + 8$
6. $x^2 - 7x + 10$
7. $a^2 - 8a + 12$
8. $c^2 - 9c + 8$
9. $p^2 - 8p + 7$
10. $x^2 - 10x + 21$
11. $a^2 + 2a - 8$
12. $b^2 + 2b - 15$
13. $k^2 - 3k - 10$
14. $x^2 + 2x - 35$
15. $r^2 - 4r - 21$
16. $c^2 - 5c - 14$
17. $x^2 + 3x - 40$
18. $a^2 - a - 20$
19. $t^2 + 2t - 63$
20. $x^2 + x - 42$
21. $4 + 5x + x^2$
22. $7 - 8a + a^2$
23. $2 + y - y^2$
24. $6 - c - c^2$
25. $8 + 2d - d^2$
26. $a^2 - a - 90$

27. $a^2 + 8a + 16$
28. $y^2 - 10y + 25$
29. $10 - 3x - x^2$.
30. $42 + 13x + x^2$
31. $a^2 + 3ab + 2b^2$
32. $x^2 - 5xy + 6y^2$
33. $p^2 - 9pq + 20q^2$
34. $k^2 + 4kl - 5l^2$
35. $x^2 + 6x + 9$
36. $x^2 - 12x + 36$
37. $a^2 - 14a + 49$
38. $36 + 12a + a^2$
39. $64 - 16x + x^2$
40. $a^2 + 2ab + b^2$
41. $x^2 - 4xy + 4y^2$
42. $2x^2 + 6x + 4$
43. $3a^2 - 15a + 12$
44. $5a^2 - 25a - 30$
45. $ax^2 + ax - 6a$
46. $kl^2 - 4kl - 12k$
47. $a^2x^2 + 2a^2x + a^2$
48. $4 - 4x + x^2$
49. $45 + 30x + 5x^2$
50. $x^3 + 5x^2 - 6x$
51. $a^3 - 12a^2 + 27a$
52. $t^4 + 5t^3 - 14t^2$
53. $a^2b^2 + 7ab + 12$
54. $x^2y^2 - 9xy + 20$
55. $2x^3 + 30x^2 + 112x$
56. $6a^3 - 30a^2 + 36a$
57. $3ax^2 - 6ax + 3a$
58. $12k - 2kx - 2kx^2$
59. $x^2 + (m + n)x + mn$
60. $a^2 - (m + n)a + mn$
61. $x^2 + (a - b)x - ab$
62. $x^2 - (a - b)x - ab$
63. $x^2 + (5a - 2b)x - 10ab$
64. $a^2 - 2a^3b + a^4b^2$
65. $x(x - 9) + 18$
66. $x^4y^2 + 2x^3y - 35x^2$
67. $x(x + 1) + 6(x - 5)$

B. Type $ax^2 + bx + c$, where a, b, c are constants.

We have already seen in the multiplication of binomials that

$$\begin{aligned}(2x - 5)(3x + 4) &= 6x^2 - 15x + 8x - 20 \\ &= 6x^2 - 7x - 20\end{aligned}$$

the terms in the answer being found as before, the middle terms in particular being found by multiplying the terms connected by the links, and adding the results.

It should be noted that

$$\begin{aligned}(-20)(6x^2) &= -120x^2 \\ (-15x)(+8x) &= -120x^2\end{aligned}$$

Example 1: Factorise $12x^2 + 8x - 15$.

Method 1. We try to find two numbers, whose sum is $+8x$ and whose product is $(+12x^2)(-15) = -180x^2$.

By inspection these numbers are $+18x$ and $-10x$.

$$\begin{aligned} \therefore\ 12x^2 + 8x - 15 &= 12x^2 + 18x - 10x - 15 \\ &= (12x^2 + 18x) - (10x + 15) \\ &= 6x(2x + 3) - 5(2x + 3) \\ &= (2x + 3)(6x - 5) \end{aligned}$$

Example 2: Factorise $6x^2 - 5x - 6$.

Method 2. We note that the factors of $6x^2$ are $2x$ and $3x$, or $6x$ and x, and that the factors of 6 are 2 and 3, or 6 and 1, also that the signs inside the brackets are unlike, since the sign of the constant term is $-$. Hence we can write down various factors that will give the first term, $6x^2$, and the last term, -6, correctly. We reject those which do not give the correct middle term, e.g. we would reject $(3x - 1)(2x + 6)$, since, although the $6x^2$ term and the -6 term are correct, the middle term $-2x + 18x$ i.e. $+16x$, is wrong. By trial it will be found that

$$6x^2 - 5x - 6 = (3x + 2)(2x - 3)$$

the middle term being $+4x - 9x = -5x$.

With experience it will be found that various common-sense considerations will shorten the work. For example,

$$(3x - 1)(2x + 6)$$

could have been rejected at once, because 2 is a common factor of $2x$ and $+6$ and, since 2 is *not* a common factor of the original expression, $(2x + 6)$ cannot be a factor.

Example 3: Factorise $16a^2 + 4ab - 6b^2$.

$$16a^2 + 4ab - 6b^2 = 2(8a^2 + 2ab - 3b^2)$$

To factorise $8a^2 + 2ab - 3b^2$ we can use either method. Using the first method, we look for two numbers whose product is $(8a^2)(-3b^2)$, i.e. $-24a^2b^2$ and whose sum is $+2ab$.

By inspection the numbers are $+6ab$ and $-4ab$.

$$\begin{aligned} \therefore\ 8a^2 + 2ab - 3b^2 &= 8a^2 + 6ab - 4ab - 3b^2 \\ &= 2a(4a + 3b) - b(4a + 3b) \\ &= (4a + 3b)(2a - b) \end{aligned}$$

$$\therefore \text{ original expression} = 2(4a + 3b)(2a - b)$$

Exercises 54

Factorise:

1. $2a^2 + 5a + 2$
2. $2x^2 + 3x + 1$
3. $3x^2 + 5x + 2$
4. $4a^2 + 4a + 1$
5. $2 + 7x + 3x^2$
6. $2a^2 - 9a + 4$
7. $2t^2 - 13t + 21$
8. $3t^2 - 17t + 10$
9. $2x^2 - 3x - 35$
10. $2x^2 + 3x - 27$
11. $2x^2 - x - 10$
12. $3a^2 - 7a - 6$
13. $3y^2 + 2y - 8$
14. $1 - x - 2x^2$
15. $3 + 8r - 3r^2$
16. $2 - 7x + 6x^2$
17. $4a^2 - 4a - 3$
18. $5a^2 + 11a + 2$
19. $6r^2 + r - 2$
20. $6r^2 + 17r - 3$
21. $6x^2 - 13x + 6$
22. $6x^2 + 5x - 6$
23. $6x^2 - 11x - 10$
24. $4a^2 - 12a - 27$
25. $5a^2 + 21a + 18$
26. $2y^2 - 25y + 12$
27. $3y^2 - 2y - 16$
28. $5y^2 + 33y - 14$
29. $5y^2 + 52y + 20$
30. $7y^2 + 23y - 20$
31. $2 + 3t - 2t^2$
32. $3 - 2x - 8x^2$
33. $5 - 11x + 6x^2$
34. $8 + 6x - 5x^2$
35. $9a^2 + 18a + 8$
36. $12p^2 + p - 6$
37. $15k^2 - 2k - 8$
38. $16x^2 - 30x + 9$
39. $20y^2 + y - 12$
40. $18a^2 + 9a - 20$
41. $2x^2 + xy - 3y^2$
42. $3x^2 - 5xy + 2y^2$
43. $4a^2 - 5ax - 6x^2$
44. $4a^2 + 15ax - 4x^2$
45. $8x^2 - 2xy - 3y^2$
46. $12x^2 - 16xy + 5y^2$
47. $16a^2 + 14ab - 15b^2$
48. $21p^2 - 5pq - 4q^2$
49. $20x^2 + 44xy + 21y^2$
50. $4y(y + 4) + 15$
51. $3(2x^2 + 7) - 23x$
52. $12p^2 + 5(4p - 5)$

53. $4x^2 - 10x - 6$

54. $4a^3 + 5a^2 - 6a$

55. $48t^3 - 12t^2 - 6t$

56. $75x^3 - 24(5x^2 - 2x)$

57. $2x^3y^3 - 4x^2y^2 - 6xy$

58. $12 - 30a + 12a^2$

59. $6x^3 - x^2y - 12xy^2$

60. $2x(4x + 7y) - 30y^2$

IV. *Difference of Squares*

$$x^2 - y^2 = (x + y)(x - y)$$

Example 1: Factorise $9a^2 - 25b^2$.

$$\begin{aligned} 9a^2 - 25b^2 &= (3a)^2 - (5b)^2 \\ &= (3a + 5b)(3a - 5b) \end{aligned}$$

Example 2: Factorise $(2x + 1)^2 - (x - 3)^2$.

$$\begin{aligned} (2x + 1)^2 - (x - 3)^2 &= [(2x + 1) + (x - 3)][(2x + 1) - (x - 3)] \\ &= (2x + 1 + x - 3)(2x + 1 - x + 3) \\ &= (3x - 2)(x + 4) \end{aligned}$$

Example 3: Factorise $9(2x - 3y)^2 - (3x + 2y)^2$

$$\begin{aligned} &9(2x - 3y)^2 - (3x + 2y)^2 \\ &= [3(2x - 3y)]^2 - (3x + 2y)^2 \\ &= [3(2x - 3y) + (3x + 2y)][3(2x - 3y) - (3x + 2y)] \\ &= (6x - 9y + 3x + 2y)(6x - 9y - 3x - 2y) \\ &= (9x - 7y)(3x - 11y) \end{aligned}$$

Example 4: Evaluate $97^2 - 93^2$.

$$\begin{aligned} 97^2 - 93^2 &= (97 + 93)(97 - 93) \\ &= (190)(4) \\ &= 760 \end{aligned}$$

Exercises 55

Factorise:

1. $x^2 - 4$

2. $a^2 - 9$

3. $16 - y^2$

4. $x^2 - 1$

5. $25 - t^2$

6. $4x^2 - 81$

7. $9a^2 - 100$

8. $a^2b^2 - 49$

9. $9x^2 - 4y^2$

10. $25k^2 - 64l^2$

11. $1 - 4x^2y^2$
12. $a^4 - b^2$
13. $a^2b^2 - c^2d^2$
14. $x^6 - y^4$
15. $\frac{1}{4}x^2 - y^2$
16. $(x + y)^2 - z^2$
17. $x^2 - (y + z)^2$
18. $a^2 - (b - c)^2$
19. $(a - b)^2 - 1$
20. $(a + b)^2 - 9$
21. $(x + 3)^2 - 16$
22. $100 - (x - 2)^2$
23. $(a + b)^2 - (c + d)^2$
24. $(a - b)^2 - (c - d)^2$
25. $(x + 5)^2 - (x - 2)^2$
26. $(2x + y)^2 - (x + 2y)^2$
27. $(x - 3y)^2 - (3x - y)^2$
28. $81 - (2a - 3)^2$
29. $9(a + b)^2 - a^2$
30. $16(x - y)^2 - 25y^2$
31. $49 - 4(x - 3y)^2$
32. $4(x + y)^2 - (x - y)^2$
33. $9(x - y)^2 - 4(x + y)^2$
34. $(3x - 2)^2 - (x + 1)^2$
35. $x^2 - 16(y - z)^2$
36. $(2x + 3y)^2 - (3x - 2y)^2$
37. $9(3a - 2b)^2 - (2a + b)^2$
38. $4(2a + 3b)^2 - 9(3a - 2b)^2$
39. $(x - y - 1)^2 - (x - 2y)^2$
40. $3x^2 - 12$
41. $ab^2 - 16a$
42. $a^4 - a^2$
43. $5 - 125c^2$
44. $2x^2 - 8a^2$
45. $75a^2 - 48b^2$
46. $12 - 108x^2$
47. $a^3 - a^5$
48. $4x^3 - 9x$
49. $4x^4 - 25y^2$
50. $a^4 - 16b^4$
51. $\pi R^2 - \pi r^2$
52. $5(x + y)^2 - 5$
53. $(x + 5)(x - 2)^2 - 4(x + 5)$
54. $x^2 - y^2 - x + y$
55. $a^2 - 9b^2 + ac + 3bc$
56. $x^2 - y^2 - yz + xz$

Evaluate:

57. $31^2 - 29^2$
58. $199^2 - 1$
59. $5{\cdot}5^2 - 4{\cdot}5^2$
60. $7{\cdot}32^2 - 7{\cdot}22^2$

Miscellaneous Factors

The following hints may help you to factorise any given algebraic expression:

1. Look for any factor common to all the terms. If one is found, write it down, and to find the other factor, divide the original expression by the common factor.

If the expression contains brackets, and no common factor is seen, the brackets should usually be removed, and the expression simplified.

2. If the expression has two terms only, it may be the difference of two squares, and the factors can then be written down.

3. If the expression is a trinomial the factors should be found by one of the methods already discussed.

4. If the expression contains four or more terms it may be possible to factorise it by grouping.

It is important at all times to remember that an expression is factorised only when it has been reduced to *one* term.

Example 1: Factorise $a(3b + 25) - 5(a^2 + 3b)$.

$$\begin{aligned} a(3b + 25) - 5(a^2 + 3b) &= 3ab + 25a - 5a^2 - 15b \\ &= (3ab - 5a^2) + (25a - 15b) \\ &= a(3b - 5a) + 5(5a - 3b) \\ &= a(3b - 5a) - 5(3b - 5a) \\ &= (3b - 5a)(a - 5) \end{aligned}$$

Example 2: Factorise $2x^2 + xy - y^2 - 6x + 3y$.

$$\begin{aligned} 2x^2 + xy - y^2 - 6x + 3y &= (2x^2 + xy - y^2) - (6x - 3y) \\ &= (2x - y)(x + y) - 3(2x - y) \\ &= (2x - y)(x + y - 3) \end{aligned}$$

Example 3: Factorise $x(x - 4) - y(y + 4)$.

$$\begin{aligned} x(x - 4) - y(y + 4) &= x^2 - 4x - y^2 - 4y \\ &= (x^2 - y^2) - (4x + 4y) \\ &= (x + y)(x - y) - 4(x + y) \\ &= (x + y)(x - y - 4) \end{aligned}$$

Example 4: Factorise $(x - 2y)^3 - 9x + 18y$.

$$\begin{aligned} &(x - 2y)^3 - 9x + 18y \\ &= (x - 2y)^3 - 9(x - 2y) \\ &= (x - 2y)[(x - 2y)^2 - 9] \\ &= (x - 2y)(x - 2y + 3)(x - 2y - 3) \end{aligned}$$

Exercises 56

Factorise:

1. $a^3 - 2a$
2. $x^2 + 2x - 63$
3. $3xy - 3ab + 9bx - ay$
4. $x^2 - xy - 42y^2$
5. $5a^2 - 125$
6. $27x^2 - 12x + 1$
7. $(x - 3) - (x - 3)(x - 4)$
8. $8a^2 + 14ab - 15b^2$
9. $3x^2 - 18x + 27$
10. $1 + ba - b - a$
11. $4x^2 + 6x + 2$
12. $x^2 - 9y^2 + x - 3y$
13. $ab^3 - a^3b$
14. $a(a - 9) - b(b - 9)$
15. $a^2b^2c^2 + 2a^2b^2c - 3a^3bc^3$
16. $x(x + 4) - y(y + 4)$
17. $12 + 11p - 15p^2$
18. $2ab - 6bc - 3ad + 9cd$
19. $6x^2 + 33x - 63$
20. $4p^2q^2 - 16$
21. $1 - 3ab - 18a^2b^2$
22. $(ax + by)^2 + (bx - ay)^2$
23. $20 - 25b + 5b^2$
24. $4a^2 - (b - c)^2$
25. $12x^3 - 27x$
26. $(3x - 2)^2 - 16$
27. $t(6t - 11) + 3$
28. $(2a - 5)^2 - (a + 3)^2$
29. $3(x + y)^2 - 2(x + y)$
30. $(x + 2y)(x - y) + x - y$
31. $a(a + 2) - 4b(b + 1)$
32. $6x^2 - xy - 2y^2 - 6x - 3y$
33. $(2a + b)^3 - 8a - 4b$
34. $18a^2b - 33ab + 12b$
35. $2xy - ax - ay + 2y^2$
36. $(3x - 2y)^2 - 16z^2$
37. $4x^3 - 13x^2y + 10xy^2$
38. $6ax - 6y - 9ay + 4x$
39. $6x(y + 1) - 4x^2 - 9y$
40. $12y^4 + 3y^2$
41. $a(a - 2) - b(b + 2)$
42. $27x(x + 1) - 2(3x + 10)$
43. $a^2 + 2a + 1 + a(a + 1)$
44. $9(2x - y)^2 - 16(x - 2y)^2$
45. $xy^2 + ax - a^2y^2 - a^3$
46. $a^5 + a(a^3 + 1) + 1$
47. $x^4 + x(x^2 + 1) - 1$
48. $a(1 - b^2) + b(1 - a^2)$
49. $x^4 - (5x + 6)^2$
50. $x(x^2 - y^2) - 2y^2(x - y)$
51. $x^3 + 7x^2 + 14x + 8$, one factor being $x + 1$
52. $x^3 - 2x^2 - 5x + 6$, one factor being $x - 3$

Chapter 19

FRACTIONS II

Reduction to Lowest Terms

In Chapter 9 we have seen that to simplify a fraction we divide the numerator and denominator by their H.C.F., e.g.

$$\frac{ab^3}{a^2b^2c^2} = \frac{ab^3 \div ab^2}{a^2b^2c^2 \div ab^2} = \frac{b}{ac^2}$$

and that this is usually shown by cancelling the common factors thus:

$$\frac{ab^3}{a^2b^2c^2} = \frac{\overset{1}{\cancel{a}}\,\overset{b}{\cancel{b^3}}}{\underset{a}{\cancel{a^2}}\,\underset{1}{\cancel{b^2}}\,c^2} = \frac{1 \times b}{a \times 1 \times c^2} = \frac{b}{ac^2}$$

In exactly the same way

$$\frac{6a^2 + 2a - 28}{a^2 - 4} = \frac{2(3a + 7)\overset{1}{\cancel{(a - 2)}}}{(a + 2)\underset{1}{\cancel{(a - 2)}}} = \frac{2(3a + 7)}{a + 2}$$

Exercises 57

Simplify:

1. $\dfrac{a^2bc^3}{a^3bc^2}$

2. $\dfrac{-12x^2y^4z}{15xy^2z^2}$

3. $\dfrac{a(b - c)}{c(b - c)}$

4. $\dfrac{(x + y)^2}{(x + y)(x - y)}$

5. $\dfrac{x - y}{(y + x)(y - x)}$

6. $\dfrac{(x + 4)(x - 2)}{(x - 2)(x - 3)}$

7. $\dfrac{3a + 15}{4a + 20}$

8. $\dfrac{8x}{x^2 + 4x}$

9. $\dfrac{5x - 10}{2x - 4}$

10. $\dfrac{ax + ay}{bx + by}$

11. $\dfrac{3x^2 - 6x}{ax - 2a}$

12. $\dfrac{ax - ab}{bx^2 - b^2x}$

13. $\dfrac{b^2x + ab}{bcx + ac}$

14. $\dfrac{x^2 + 5x + 6}{x^2 + 3x + 2}$

15. $\dfrac{x^2 + x - 6}{x^2 + 2x - 3}$

16. $\dfrac{x^2 + 2x - 24}{x^2 + 4x - 32}$

17. $\dfrac{x^2 - 10x + 21}{x^2 - 6x + 9}$

18. $\dfrac{a^2 - b^2}{a^2 + 2ab + b^2}$

19. $\dfrac{k^2 + 4kl - 5l^2}{k^2 - l^2}$

20. $\dfrac{2x^2 + 7x + 6}{6x^2 + 11x + 3}$

21. $\dfrac{3x^2 + x - 2}{3x^2 - x - 4}$

22. $\dfrac{15a^2 - 14a - 8}{9a^2 - 16}$

23. $\dfrac{x^2 + 7x}{2x^2 - 98}$

24. $\dfrac{a^2 + 3ab}{5a + 15b}$

25. $\dfrac{x^2 - 16}{x^2 - 4x}$

26. $\dfrac{4 + 7x - 2x^2}{4 - 5x + x^2}$

27. $\dfrac{a^2 + ab - 12b^2}{2a^2 + 9ab + 4b^2}$

28. $\dfrac{8x^2 - 2xy - 15y^2}{16x^2 - 25y^2}$

29. $\dfrac{2a^3 + 3a^2b - 5ab^2}{2a^2 + 7ab + 5b^2}$

30. $\dfrac{12x^2 - xy - 6y^2}{8x^2 + 14xy - 15y^2}$

31. $\dfrac{(a + b)^2 - c^2}{a^2 - (b - c)^2}$

32. $\dfrac{3ax - 5ay - 3bx + 5by}{a^2 - b^2 + (a - b)^2}$

33. $\dfrac{2lm - 6n^2 - 4mn + 3ln}{6m^2 + 13mn + 6n^2}$

34. $\dfrac{(2p + 3q)^2 - (3p + 2q)^2}{5pq - 2p^2 - 3q^2}$

35. $\dfrac{2b(a - 3b) + 5a(a + b)}{3b(b - 3a) + 2a(5a - b)}$

36. $\dfrac{2a - a^2 + 2b - ab}{2ab - 4(a + b) + 2a^2}$

Multiplication and Division

Example: Simplify:

$$\frac{6x+3y}{4x^2-y^2} \times \frac{2x^2+3xy-2y^2}{3x} \div \frac{x^2+4xy+4y^2}{2x^3-x^2y}$$

We factorise the numerators and denominators and then cancel common factors.

Expression

$$= \frac{\overset{1}{\cancel{3}}\overset{1}{\cancel{(2x+y)}}}{\underset{1}{\cancel{(2x-y)}}\underset{1}{\cancel{(2x+y)}}} \times \frac{\overset{1}{\cancel{(2x-y)}}\overset{1}{\cancel{(x+2y)}}}{\underset{1\ 1}{\cancel{3x}}} \times \frac{\overset{x}{\cancel{x^2}}(2x-y)}{\underset{x+2y}{\cancel{(x+2y)^2}}}$$

$$= \frac{x(2x-y)}{x+2y}$$

Exercises 58

1. $\dfrac{16a^3b^2}{12b^3c^2} \times \dfrac{3c^3}{8a^2}$

2. $\dfrac{-6x^2}{5y} \times \dfrac{y^2}{4x} \div \dfrac{3x}{y}$

3. $\dfrac{2a+8}{bc} \times \dfrac{c^2}{a^2+4a}$

4. $\dfrac{ab+a^2}{a^2} \div \dfrac{ab+b^2}{b^2}$

5. $\dfrac{x^2-y^2}{xy-y^2} \times \dfrac{yz}{xz+yz}$

6. $\dfrac{x^2+8x+15}{x^2-9} \times \dfrac{x^2-3x}{xy+5y}$

7. $\dfrac{x^2-5x+6}{4x-12} \div \dfrac{x^2-x-2}{2x^2-2}$

8. $\dfrac{x^2-4x+4}{x^2+3x} \times \dfrac{x^2+6x+9}{x^2+x-6}$

9. $\dfrac{c^2-bc}{ab} \div \dfrac{ac-ab}{b}$

10. $\dfrac{a^3+ab^2}{2a^2+9ab-5b^2} \div \dfrac{a^2+b^2}{3a+15b}$

11. $\dfrac{2x^2+6x}{2x^2} \times \dfrac{x-3}{x} \times \dfrac{x^2}{x^2-9}$

12. $\dfrac{a-b}{a-c} \times \dfrac{ab}{b-a} \times \dfrac{c-a}{bc}$

13. $\dfrac{x^2+x-2}{2x} \times \dfrac{x^2+2x+1}{x^2+3x+2}$

14. $\dfrac{3b-3c}{a^2-2ab+b^2} \times \dfrac{c}{b} \div \dfrac{bc-c^2}{ab-b^2}$.

15. $\dfrac{1+t^2}{2t} \div \dfrac{1-t^2}{2t^2} \times \dfrac{1+t}{t}$.

16. $\dfrac{x^2-5x+6}{x^2-9x+18} \div \dfrac{x-2}{ax^2-6ax}$.

17. $\dfrac{x^2-x-20}{x^2+2x-15} \times \dfrac{2x^2-50}{6x} \times \dfrac{3x}{x^2-10x+25}$

18. $\dfrac{2x^2+4xy}{x^2+xy-2y^2} \div \dfrac{2x^3-6x^2y}{x^2-4xy+3y^2}$.

19. $\dfrac{9x^2-y^2}{12x+4y} \times \dfrac{3x^2+8xy-3y^2}{3x^2-xy}$.

20. $\dfrac{y}{x+y} \times \dfrac{x^2-y^2}{x^2y} \div \left(\dfrac{1}{y}-\dfrac{1}{x}\right)$.

21. $\dfrac{6-5x+x^2}{4-x^2} \times \dfrac{2-5x-3x^2}{x^2} \div \dfrac{6+x-x^2}{x}$.

L.C.M.—Addition and Subtraction of Fractions

Example 1: Find the L.C.M. of

$$2a^2+6a,\ a^2-9,\ a^2-6a+9$$

$$\begin{aligned} 2a^2+6a &= 2a(a+3) \\ a^2-9 &= (a-3)(a+3) \\ a^2-6a+9 &= (a-3)^2 \\ \therefore \text{L.C.M.} &= 2a(a+3)(a-3)^2 \end{aligned}$$

Example 2: Simplify $\dfrac{3x}{4x-16} - \dfrac{x+8}{x^2-4x} + \dfrac{1}{x}$.

As before, we factorise the denominators, find the L.C.M. and rewrite each fraction with the L.C.M. as its denominator.

$$\begin{aligned} \text{Expression} &= \frac{3x}{4(x-4)} - \frac{x+8}{x(x-4)} + \frac{1}{x} \\ &= \frac{3x(x)}{4x(x-4)} - \frac{(x+8)(4)}{4x(x-4)} + \frac{1(4)(x-4)}{4x(x-4)} \end{aligned}$$

$$\text{Expression} = \frac{3x^2}{4x(x-4)} + \frac{4(x+8)}{4x(x-4)} + \frac{4(x-4)}{4x(x-4)}$$
$$= \frac{3x^2 - 4x - 32 + 4x - 16}{4x(x-4)}$$
$$= \frac{3x^2 - 48}{4x(x-4)}$$
$$= \frac{3(x^2 - 16)}{4x(x-4)}$$
$$= \frac{3(x-4)(x+4)}{4x(x-4)}$$
$$= \frac{3(x+4)}{4x}$$

Exercises 59

Find the L.C.M. of:

1. $a^2b,\ ab^2$
2. $x,\ x(x+2)$
3. $2(x+3)^2,\ 6(x+3)$
4. $2a-4,\ 8a-16$
5. $a,\ a^2-a$
6. $6t^2+10t,\ 2t$
7. $x^2+x,\ 3x+3$
8. $x^2-xy,\ xy-y^2$
9. $4t-8,\ 6t-12$
10. $x^2-5x+6,\ x^2+2x-8$
11. $x^2+3x+2,\ x^2-3x-4$
12. $x^2-9,\ x^2+5x+6$
13. $x^2-4,\ x^2-2x$
14. $3x,\ x-2,\ x^2-3x+2$
15. $a^2-25,\ 2a-10,\ 4a+20$
16. $1-x,\ 2+2x,\ 1-x^2$
17. $x^2+5x+6,\ (x+2)^2,\ x^2+3x$
18. $x^2-2x+1,\ x^2-1,\ x^2+x$
19. $x^4-1,\ x^2-1,\ x+1$
20. $4x^2-9y^2,\ 6x^2+5xy-6y^2,\ 6x^2-13xy+6y^2$

Exercises 60

Simplify:

1. $\dfrac{3x}{4} - \dfrac{2x}{3}$

2. $\dfrac{2}{3p} + \dfrac{3}{4p}$

3. $\dfrac{1}{d} + \dfrac{3}{2d} - \dfrac{2}{5d}$

4. $\dfrac{3}{ab} - \dfrac{2}{a} + 1$

5. $1 + \dfrac{1}{\dfrac{1}{a}}$

6. $\dfrac{2x + 3y}{6} - \dfrac{x - 2y}{2}$

7. $\frac{3}{4}(2x - 1) - \frac{2}{3}(2x + 1)$

8. $\dfrac{a - b}{ab} + \dfrac{b - c}{bc}$

9. $1 + \dfrac{1}{a - 1}$

10. $1 - \dfrac{1}{1 - \dfrac{1}{a}}$

11. $\dfrac{1}{1 + \dfrac{1}{z}} - z$

12. $\dfrac{1}{a + 1} - 3$

13. $\dfrac{a}{a - b} - 1$

14. $\dfrac{2}{t + 1} - t$

15. $\dfrac{3}{t - 1} - \dfrac{1}{t}$

16. $b - \dfrac{ab}{a + b}$

17. $\dfrac{1}{x + 2} + \dfrac{1}{x + 3}$

18. $\dfrac{3}{x - 3} - \dfrac{2}{x + 2}$

19. $\dfrac{x}{x - y} - \dfrac{y}{y - x}$

20. $x - \dfrac{xy}{y - x}$

21. $\dfrac{k + 2}{4} - \dfrac{k}{5}$

22. $\dfrac{2k - 3}{6} - \dfrac{3k + 1}{8}$

23. $y + \dfrac{2y - 3}{3} - \dfrac{y - 4}{4}$

24. $\dfrac{2}{x(x - 2)} - \dfrac{1}{x - 2}$

25. $\dfrac{5}{a^2 + 5a} - \dfrac{1}{a}$

26. $\dfrac{p + q}{p - q} - \dfrac{p - q}{p + q}$

27. $\dfrac{1}{y} + \dfrac{2y}{x^2 - y^2}$

28. $\dfrac{a}{a^2 - ab} - \dfrac{1}{b}$

29. $\dfrac{2a + 6}{a^2 - 9} - \dfrac{3}{a + 3}$

30. $\dfrac{a}{ab + b^2} - \dfrac{b}{a^2 + ab}$

31. $\dfrac{1}{x^2 - 4x - 21} - \dfrac{1}{x^2 + 2x - 63}$

32. $\dfrac{2}{x^2 - 2x - 15} + \dfrac{1}{x^2 + x - 6}$

33. $\dfrac{3}{2 - 5x + 3x^2} + \dfrac{4}{2 - x - 3x^2}$

34. $\dfrac{3}{x^2 + 4x - 5} + \dfrac{2}{x^2 - 8x + 7}$

35. $\dfrac{t + 3}{t^2 + 2t - 3} + \dfrac{t - 4}{t^2 - 16}$

36. $\dfrac{3}{x - 3} - \dfrac{5x + 6}{x^2 + x - 12}$

37. $\dfrac{1}{x^2 + 2xy + y^2} - \dfrac{1}{x^2 - y^2}$

38. $\dfrac{3}{x^2 + 2xy - 3y^2} + \dfrac{2}{x^2 - 3xy + 2y^2}$

39. $\dfrac{2}{9x^2 - 3xy - 2y^2} + \dfrac{1}{9x^2 - 4y^2}$

40. $\dfrac{2}{3 - 2a - 8a^2} - \dfrac{4}{6 - a - 12a^2}$

41. $\dfrac{1}{x^2 + x - 2} + \dfrac{3}{x^2 - x - 6} - \dfrac{3}{x^2 - 4x + 3}$

42. $1 + \dfrac{3}{x} + \dfrac{12}{x(x - 4)}$

43. $\dfrac{2}{x} - \dfrac{4}{x(x + 2)} - \dfrac{1}{x + 2}$

44. $\dfrac{3a}{a^2 - 4} - \dfrac{2}{a + 2} - \dfrac{1}{a - 2}$

45. $\dfrac{3}{k + 1} - \dfrac{2}{k - 2} + \dfrac{1}{k - 3}$

46. $\dfrac{a - 2}{a^2 + 2a} - \dfrac{a + 4}{a^2 + 5a + 6} + \dfrac{3}{a^2 + 3a}$

47. $\dfrac{1}{1 - x} + \dfrac{1}{1 + x} - \dfrac{1}{1 + x^2}$

48. $\dfrac{2}{6 + a - a^2} - \dfrac{1}{(2 + a)^2} + \dfrac{3}{9 - 6a + a^2}$

49. $\dfrac{1}{x}\left(1 - \dfrac{x}{y}\right) + \dfrac{1}{y}\left(1 - \dfrac{y}{z}\right) - \dfrac{1}{z}\left(1 - \dfrac{z}{x}\right)$.

50. $\left(\dfrac{a}{b} - \dfrac{b}{a}\right) \div \left(\dfrac{1}{a} + \dfrac{1}{b}\right) + \dfrac{4ab}{a - b}$

Exercises 61

Prove:

1. $\dfrac{1}{x - y} - \dfrac{1}{x + y} = \dfrac{2y}{x^2 - y^2}$

2. $\dfrac{a}{x} + \dfrac{x}{a} - 1 = \dfrac{(a - x)^2}{ax} + 1$

3. $\dfrac{x + y}{x - y} - \dfrac{x - y}{x + y} = \dfrac{4xy}{x^2 - y^2}$

4. $\dfrac{x}{x - a} + \dfrac{x}{x + a} - 2 = \dfrac{2a^2}{x^2 - a^2}$

5. $\dfrac{1}{2x + 4} - \dfrac{1}{3x - 6} + \dfrac{x + 8}{3(x^2 - 4)} = \dfrac{1}{2x - 4}$

6. $\dfrac{x}{xy - y^2} - \dfrac{y}{x^2 - xy} - \dfrac{1}{x} = \dfrac{1}{y}$

7. $\dfrac{a - b}{ab} + \dfrac{b - c}{bc} = \dfrac{1}{c} - \dfrac{1}{a}$

8. $\left(\dfrac{1}{x} - x\right)\left(\dfrac{1}{x} + x\right) \div \left(\dfrac{1}{2x} - \dfrac{x^3}{2}\right) = \dfrac{2}{x}$

9. $\dfrac{b}{a^2 - ab} - \dfrac{a}{ab - b^2} + \dfrac{a^2 + b^2}{ab(a - b)} = \dfrac{2b}{a^2 - ab}$

10. $\dfrac{a}{(a - b)(c - a)} + \dfrac{b}{(b - c)(a - b)} + \dfrac{c}{(c - a)(b - c)} = 0$

11. $\left(\dfrac{1}{p + q} - \dfrac{1}{q}\right)\left(\dfrac{1}{p - q} - \dfrac{1}{p}\right)\left(\dfrac{1}{p^2} - \dfrac{1}{q^2}\right) = \dfrac{1}{p^2q^2}$

12. $\dfrac{x - c}{(x - a)(x - b)} + \dfrac{b - c}{(a - b)(x - b)} = \dfrac{a - c}{(a - b)(x - a)}$

Simple Equations Involving Fractions

Example 1: Solve $\frac{4}{x+1} = \frac{3}{x-2}$.

$$\frac{4}{x+1} = \frac{3}{x-2}$$

Multiply both sides by the L.C.M. of the denominators, i.e. $(x+1)(x-2)$.

$$\therefore \frac{4}{x+1} \times (x+1)(x-2) = \frac{3}{x-2} \times (x+1)(x-2)$$

$$\therefore 4(x-2) = 3(x+1)$$

$$\therefore 4x - 8 = 3x + 3$$

$$4x - 3x = 8 + 3$$

$$\therefore x = 11$$

Example 2: Solve $\frac{x+3}{3x-4} + \frac{x-1}{2x+3} = \frac{5x^2-3}{6x^2+x-12}$.

$$\frac{x+3}{3x-4} + \frac{x-1}{2x+3} = \frac{5x^2-3}{(3x-4)(2x+3)}$$

The L.C.M. of the three denominators is $(3x-4)(2x+3)$.
Multiply both sides by this expression.

$$\therefore \frac{x+3}{3x-4} \times (3x-4)(2x+3) + \frac{x-1}{2x+3} \times (3x-4)(2x+3)$$

$$= \frac{5x^2-3}{(3x-4)(2x+3)} \times (3x-4)(2x+3)$$

$$(x+3)(2x+3) + (x-1)(3x-4) = 5x^2 - 3$$

$$\therefore 2x^2 + 9x + 9 + 3x^2 - 7x + 4 = 5x^2 - 3$$

$$\therefore 2x^2 + 9x + 3x^2 - 7x - 5x^2 = -3 - 9 - 4$$

$$\therefore 2x = -16$$

$$\therefore x = -8$$

Note: If we are given an equation with only *one* term, in the form of a fraction, on each side, e.g. $\frac{A}{B} = \frac{C}{D}$ it should be

noticed that the L.C.M. of the denominators is BD, and that if we multiply both sides by BD we get

$$\frac{A}{B} \times BD = \frac{C}{D} \times BD$$
$$\text{i.e. } AD = BC$$

This same result can be obtained from the original equation by carrying out the multiplication indicated thus:

$$\frac{A}{B} \times \frac{C}{D}$$
$$\text{i.e. } AD = BC$$

This is called *cross-multiplication*, and using this we can often shorten the working. The greatest care must be taken to use cross-multiplication only when there is a single term on each side of the equation, e.g.:

$$\text{Solve } \frac{x-2}{x+3} = \frac{x-3}{x+7}$$

By cross-multiplication

$$\therefore\ (x-2)(x+7) = (x+3)(x-3)$$
$$\therefore\ x^2 + 5x - 14 = x^2 - 9$$
$$\therefore\ x^2 + 5x - x^2 = -9 + 14$$
$$\therefore\ 5x = 5$$
$$\therefore\ x = 1$$

Exercises 62

Solve:

1. $\frac{1}{x+2} = 3$
2. $\frac{4}{x-3} = 2$
3. $\frac{2}{x-2} - 6 = 0$
4. $\frac{9}{3-x} + 3 = 0$

5. $\dfrac{3x}{5x-4} = 3$

6. $\dfrac{2x+5}{3x} = \frac{1}{2}$

7. $1\frac{2}{3} - \dfrac{4}{3-2x} = 0$

8. $\dfrac{4}{x} = \dfrac{5}{3x} + 2\frac{1}{3}$

9. $\dfrac{4}{5x} - \dfrac{2}{3(1+x)} = 0$

10. $\dfrac{3}{1+x} = \dfrac{5}{x+3}$

11. $\dfrac{x+1}{2x+3} = \dfrac{x}{2x+5}$

12. $\dfrac{x-1}{2x+1} - \dfrac{2x-3}{4x+1} = 0$

13. $\dfrac{2}{x} + \dfrac{3x}{x-1} = 3$

14. $\dfrac{x}{x-2} + \dfrac{x}{x+3} = 2$

15. $\dfrac{2}{x-2} - \dfrac{4}{x^2-4} = \dfrac{1}{x+2}$

16. $\dfrac{1}{2x+3} + \dfrac{1}{2x-1} = \dfrac{1}{4x^2+4x-3}$

17. $\dfrac{3}{y} - \dfrac{2}{y+3} = \dfrac{y}{y^2-9}$

18. $\dfrac{1}{t-1} + \dfrac{2}{t-2} - \dfrac{3}{t+3} = 0$

19. $\dfrac{x}{2x-4} - \dfrac{x+2}{3x+3} = \dfrac{x^2}{6(x-2)(x+1)}$

20. $\dfrac{u+1}{3} - \dfrac{u}{2} = \dfrac{u^2}{3(u-4)} - \dfrac{u-2}{2}$

CHAPTER 20

QUADRATIC EQUATIONS AND PROBLEMS I—SOLUTION BY FACTORS

WE have already seen how to solve simple equations, i.e. equations such as $\frac{3+x}{4} = \frac{1-3x}{3}$, etc., in which the unknown number appears to the first power.

Equations such as $x^2 - 3x = 0$ and $2x^2 + x = 1$, in which the unknown number appears to the second power and to no higher power, are called quadratic equations.

Before proceeding to solve these equations by factors we must note the following:

x and $(x - 1)$ are unknown numbers. If, however, we are told that their product is zero, i.e. $x(x - 1) = 0$, we know that one of these numbers must be zero, for, if the product of any two numbers is zero, one of them must be zero.

Hence, if $x(x - 1) = 0$, either $x = 0$ or $x - 1 = 0$.

Example 1: Solve $x^2 - 3x + 2 = 0$.

$$x^2 - 3x + 2 = 0$$
$$\therefore\ (x - 1)(x - 2) = 0$$
$$\therefore\ \text{either } x - 1 = 0 \text{ or } x - 2 = 0$$
$$\therefore\ \text{either } x = 1 \text{ or } x = 2$$

The *solution* of the equation is $x = 1$ or $x = 2$.
1 and 2 are said to be the *roots* of the equation.

Example 2: Solve $x^2 - 2x = 0$.

$$x^2 - 2x = 0$$
$$\therefore\ x(x - 2) = 0$$
$$\therefore\ \text{either } x = 0 \text{ or } x - 2 = 0$$
$$\therefore\ \text{either } x = 0 \text{ or } x = 2$$

Example 3: Solve $(2x + 5)(3x - 4) = -17$.

$$(2x + 5)(3x - 4) = -17$$
$$\therefore\ 6x^2 + 7x - 20 = -17$$
$$\therefore\ 6x^2 + 7x - 3 = 0$$
$$\therefore\ (3x - 1)(2x + 3) = 0$$
$$\therefore\ \text{either } 3x - 1 = 0 \text{ or } 2x + 3 = 0$$
$$\therefore\ \text{either } x = \tfrac{1}{3} \text{ or } x = -\tfrac{3}{2}$$

In Examples 1–3 each equation has two distinct roots, and, in general, this is the case with a quadratic equation.

An equation of the first degree in x has one root.

An equation of the second degree in x, i.e. a quadratic equation, has two roots.

Consider the following quadratic equation:

Example 4: Solve $x^2 + 8x = -16$.

$$x^2 + 8x = -16$$
$$\therefore\ x^2 + 8x + 16 = 0$$
$$\therefore\ (x + 4)(x + 4) = 0$$
$$\therefore\ x + 4 = 0 \text{ or } x + 4 = 0$$
$$\therefore\ x = -4$$

Here there is only one value of x which satisfies the equation. We say that this equation has two *equal* roots or that the root -4 is *repeated* and thus this solution is written $x = -4, -4$.

Example 5: Form the equation whose roots are 2 and $-\frac{3}{2}$.

$$x = 2 \text{ or } x = -\tfrac{3}{2}$$
$$\text{i.e. } x = 2 \text{ or } 2x = -3$$
$$\therefore\ x - 2 = 0 \text{ or } 2x + 3 = 0$$
$$\therefore\ (x - 2)(2x + 3) = 0 \text{ is the equation}$$
$$\text{i.e. } 2x^2 - x - 6 = 0 \text{ is the equation}$$

Exercises 63

Solve:

1. $(x - 2)(x - 3) = 0$
2. $(x + 2)(x + 1) = 0$
3. $(x - 4)(x + 3) = 0$
4. $(x + 5)(x - 2) = 0$

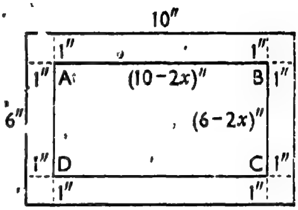
5. $x(x - 1) = 0$
6. $5x(x + 3) = 0$
7. $(2 - x)(3 - x) = 0$
8. $2(7 + x)(2 + x) = 0$
9. $x^2 = 0$
10. $x^2 = 1$
11. $4x^2 = 0$
12. $(x - 7)^2 = 0$
13. $(x + 5)^2 = 0$
14. $4(x + 2)^2 = 0$
15. $(3y - 1)(2y - 5) = 0$
16. $3(2y + 3)(y - 4) = 0$
17. $(5y + 2)(2y + 5) = 0$
18. $z^2 = 4$
19. $z^2 - z = 0$
20. $2z^2 + 2z = 0$
21. $x^2 - 25 = 0$
22. $3x^2 - 27 = 0$
23. $y^2 - 3y + 2 = 0$
24. $y^2 - 8y + 15 = 0$
25. $x^2 + x - 6 = 0$
26. $x^2 - 3x - 10 = 0$
27. $30 - 11t + t^2 = 0$
28. $24 - 5t - t^2 = 0$
29. $9 + 6t + t^2 = 0$
30. $2y^2 - 5y - 3 = 0$
31. $3y^2 - 11y + 6 = 0$
32. $24y^2 + 14y = 24$
33. $4x^2 - 16x = 0$
34. $30 = 5x^2 - 5x$
35. $6x^2 = x + 35$
36. $9 - 6x = 8x^2$
37. $6r^2 + 2r = 48$
38. $20x^2 + 51x + 28 = 0$
39. $4x^2 = x(x + 2) + 16$
40. $2(x + 1)^2 = 2(x + 7) + 100$
41. $2x(3x + 1) + 3x(2x + 1) = 2$
42. $2x(x - 1) + 2x = 33 - x(x - 2)$
43. $x(x - 3) + x(x - 4) = x(x + 4) - 24$
44. $(2x + 3)(3x - 2) = (3 - x)(1 + x) - 5$
45. $(1 - 5x)(3 + 2x) = (2 - x)^2 - 3$
46. $\dfrac{x}{2} = \dfrac{2}{x}$
47. $\dfrac{x}{2} = \dfrac{3}{2x - 1}$
48. $\dfrac{x}{x + 4} = \dfrac{x - 4}{6}$
49. $\dfrac{x + 3}{x - 1} = \dfrac{3(x + 1)}{2x - 1}$
50. $x + \dfrac{3}{x} = 4$

Find the quadratic equations whose roots are:

51. 1, 2
52. −2, −3
53. 5, −2
54. −3, 4
55. 0, 2
56. 0, −3

57. $\frac{1}{3}, \frac{1}{3}$ **58.** $-\frac{1}{2}, -\frac{1}{3}$
59. $\frac{2}{3}, -\frac{3}{4}$ **60.** $-\frac{4}{5}, 1\frac{1}{3}$

Problems Involving Quadratic Equations



Example: A rectangular sheet of cardboard measures 10 in. by 6 in. This is made into a box by cutting out squares from the corners and turning up the edges. If the area of the bottom of the box thus formed is 32 sq. in., find the length of the side of the square cut out.

Let x in. = length of the side of the square cut out.

Then

Length of bottom of box (AB) = $(10 - 2x)$ in

Breadth „ „ (CB) = $(6 - 2x)$ in

$\therefore$ Area of bottom of box = $(10 - 2x)(6 - 2x)$ sq. in.

$$\therefore (10 - 2x)(6 - 2x) = 32$$
$$\therefore 60 - 32x + 4x^2 = 32$$
$$\therefore 4x^2 - 32x + 28 = 0$$
$$\therefore x^2 - 8x + 7 = 0$$
$$\therefore (x - 7)(x - 1) = 0$$
$$\therefore \text{either } x = 7 \text{ or } x = 1,$$

But x cannot be 7, since the breadth of the cardboard is 6 in.

$$\therefore x = 1$$

Length of side of square cut out is 1 in.

Exercises 64

1. Divide 23 into two parts such that their product is 132.
2. Two numbers differ by 4. Their product is 117. Find them.
3. The product of two consecutive numbers is 182. Find them.
4. The product of two consecutive odd numbers is 195. Find them.

5. What number exceeds its positive square root by 42?

6. A man buys x articles at $(x - 2)$ shillings each. He spends £18 altogether. Find x.

7. If, to the square of a certain positive whole number, we add five times the number, the result is 84. Find the number.

8. The length of a rectangle is $(x + 3)$ in. and its breadth $(x - 3)$ in. If the area is 160 sq. in., find the length and the breadth.

9. The area of a square whose side measures $(2x + 3)$ in. exceeds the area of a rectangle $(3x - 2)$ in. by $(x + 4)$ in. by 80 sq. in. Find x.

10. A rectangular garden is twice as long as it is broad, and consists of a grass plot surrounded by a path 1 yd. wide. If the area of the grass plot is 220 sq. yd., find the dimensions of the garden.

11. Use Pythagoras' Theorem to find by calculation the lengths of the sides of a right-angled triangle if the hypotenuse is 25 in. long and the sum of the other two sides is 31 in.

12. The perimeter of a rectangle is 21 in. The diagonal measures 7·5 in. Find the dimensions of the rectangle.

13. A rectangular sheet of cardboard measures 10 in. by 8 in. Squares are cut out of the corners, and the edges turned up to make a box. If the area of the bottom of the box is 63 sq. in., find the area of the cardboard cut away.

14. The sum of the products of three consecutive positive integers taken two at a time is 74. Find the integers.

15. The length of a certain rectangle is double its breadth. Its area exceeds by 14 sq. in. the area of another rectangle half as broad as the first but 1 in. longer. Find the dimensions of the first rectangle.

16. The present ages of a man and his son are 38 years and 12 years respectively. What age was the father when the product of their ages was 87?

17. A plane figure bounded by n sides has $\frac{n(n-3)}{2}$ diagonals. If such a figure has 20 diagonals, how many sides has it?

18. A man bought $2x$ lb. of tea at $(x-4)$ shillings per lb. He sold half of it at $7s.$ per lb. and the other half at $(x-3)$ shillings per lb. He made a profit of £1 $7s.$ How many lb. of tea did he buy?

19. Divide a straight line 20 in. long into two parts so that the sum of the squares on the two parts is 218 sq. in.

20. If a stone is thrown vertically upwards from the ground with a velocity of u ft. per sec., its distance (in feet) above the ground, after t sec., is given by the expression $ut - 16t^2$. Find t when $u = 160$ and the distance above the ground is 256 ft.

21. Using the formula in Exercise 20, find at what times a body, thrown vertically upwards with a velocity of 80 ft. per sec. is 84 ft. above the ground.

22. The formula $\frac{n(n+1)}{2}$ gives the sum of the first n integers. How many integers must we take for their sum to be 78?

23. A man walked a distance of 12 miles at x m.p.h. How many hours did he take? If he had walked 1 m.p.h. slower, he would have taken 1 hr. longer.

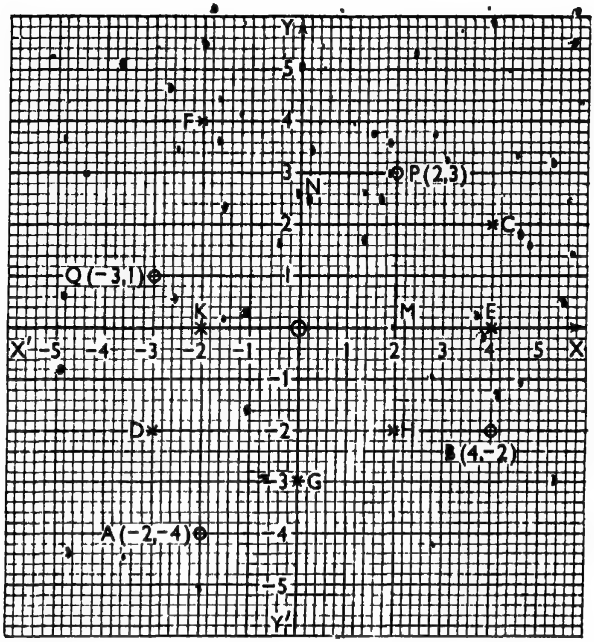
Find his slower rate of walking.

24. In a football league each team plays every other team twice in a season. How many teams were there in the league if the total number of games played in one season was 210?

25. A man bought a square plot of ground at $5s.$ per square yard. He fenced it at a cost of $4d.$ per foot. His total bill was £104. Find the area of the plot in square yards.

CHAPTER 21

GRAPHS II—THE STRAIGHT LINE



CONSIDER the above diagram on squared paper. Two straight lines X′OX and Y′OY have been drawn intersecting at right angles. The position of any point on the diagram is known if its distance from each of these two straight lines is known, e.g., the point P is 2 units from Y′OY and 3 units from X′OX. A scale must be chosen for X′OX and also for Y′OY. In this case the two scales are the same, namely 1 unit = $\frac{1}{2}$ in., but this need not be the case. Distances along X′OX measured to the right of O are reckoned positive; distances to the left of O are reckoned negative. Similarly, distances along Y′OY measured up from O are reckoned

positive; distances down from O are reckoned negative. X'OX and Y'OY are called axes of reference, or briefly *axes*. X'OX is the x-axis, Y'OY the y-axis. O is called the *origin*. The distances of any point from the axes are called the *co-ordinates* of the point, e.g. NP (or OM), the distance of P from the y-axis is called the *x co-ordinate* or the *abscissa* of P. MP (or ON), the distance of P from the x-axis is called the *y co-ordinate* or the *ordinate* of P.

P is said to be the point (2, 3), the two co-ordinates of the point being written within brackets, the x co-ordinate being written first. To fix the position of the point from its co-ordinates (2, 3) we start from the origin O, take 2 steps along OX to the point M, and then 3 steps in a direction parallel to OY, reaching the point P. This process is called *plotting* the point P.

Similarly, Q is the point $(-3, 1)$, A is the point $(-2, -4)$, and B is the point $(4, -2)$.

The axes divide the page into 4 *quadrants*, the top right being the 1st quadrant, and the others numbered thereafter 2nd, 3rd and 4th proceeding from the first in a counter-clockwise direction. Hence P and C are in the 1st quadrant, F and Q in the 2nd, D and A in the 3rd, H and B in the 4th.

Exercises 65

1. From the diagram read off the co-ordinates of the points C, D, E, F, G, H, O, K.

2. Draw, on squared paper, the x and y axes, and mark the following points: (3, 2), $(4, -3)$, (0, 2), $(-2, 0)$, $(-4, -1)$, $(-3, 3)$.

3. Find, by measurement, correct to one decimal place, the length in units of the straight line joining each pair of points:

(i) (4, 3) and $(-2, 5)$ (iii) $(-2, 3)$ and $(2, -3)$
(ii) (3, 1) and $(-1, -3)$ (iv) $(0, -3)$ and $(-4, 0)$

What are the co-ordinates of the mid-point of each line?

4. Plot the following points: (2, 0), (2, −2), (2, 3), (2, −5). What do you notice?

5. Plot the following points: (3, −4), (−3, −4), (0, −4), (1, −4). What do you notice?

6. What do you notice about the co-ordinates of the following points: (0, 0), (2, 2), (−1, −1), (5, 5)? Plot them. What do you notice?

7. Plot the following points: (1, 5), (−1, −1), (4, 14), (−3, −7), (0, 2). What do you notice?

8. Find the co-ordinates of the point of intersection of the straight lines joining the following points: (*a*) (4, 1) and (−1, 6), (*b*) (0, 2) and (−3, −4).

---

1. The points (1, 2), (2, 4), (3, 6), (−1, −2), (−2, −4), (−3, −6) are plotted on the diagram on p. 202. They all lie on a straight line (the continuous line) which passes through the origin. If we take any point on this straight line we find that its y co-ordinate is twice its x co-ordinate, in other words its co-ordinates satisfy the equation $y = 2x$. Also if we take the co-ordinates of any point which satisfy this equation, e.g. $(1\frac{1}{2}, 3)$, the point is found to lie on this line. If the co-ordinates of any point do not satisfy this equation, e.g. (1, 3), then it will be found that the point does not lie on the line. We say that $y = 2x$ is the equation of the line, and that this line is the graph of the equation $y = 2x$.

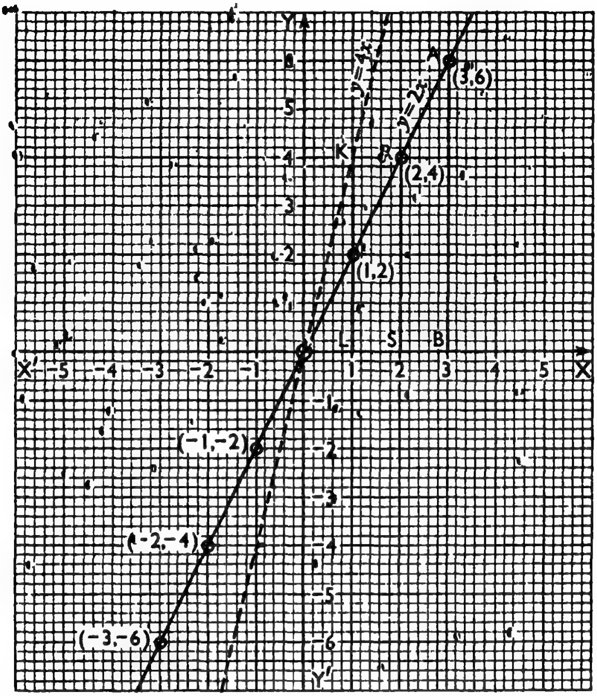
Consider the right-angled triangles AOB, ROS.

$$\text{The ratio } \frac{BA}{OB} = \frac{y \text{ co-ordinate of A}}{x \text{ co-ordinate of A}} = \frac{6}{3} = \frac{2}{1}$$

$$\text{The ratio } \frac{SR}{OS} = \frac{y \text{ co-ordinate of R}}{x \text{ co-ordinate of R}} = \frac{4}{2} = \frac{2}{1}$$

This ratio is a constant for this line. We call it the slope or *gradient* of the line—in this case it is 2.

[From the fact that AOB and ROS are similar triangles we could have deduced at once that $\frac{BA}{OB} = \frac{RS}{OS}$.]



II. If we consider the dotted line on the diagram and take the point K, we find that the gradient of this line

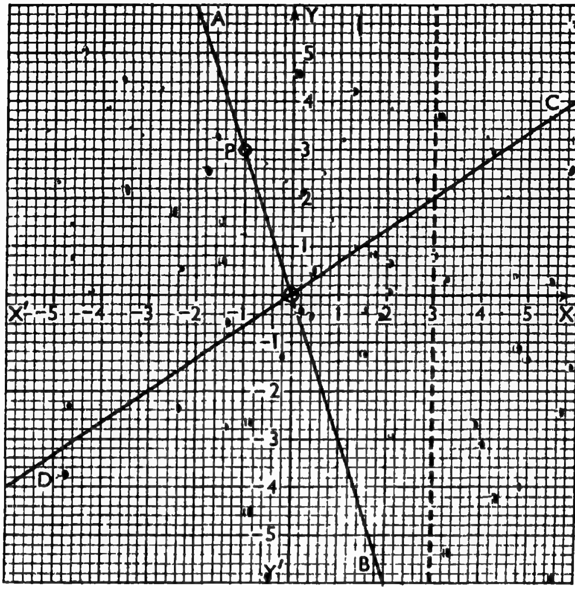
$$= \frac{LK}{OL} = \frac{4}{1} = \frac{y \text{ co-ordinate of K}}{x \text{ co-ordinate of K}}$$

This is true for any point on the line.

Hence for any point on the line

the y co-ordinate $= 4$ times the x co-ordinate

i.e. $y = 4x$ is the equation of this line.



III. In the above diagram what are the co-ordinates of the point P on the straight line AB?

Find the value of the ratio $\frac{y \text{ co-ordinate of P}}{x \text{ co-ordinate of P}}$.

What is then the equation of the straight line AB?
Without plotting the points answer the next two questions.

(1) Does the point $(-2, 5)$ lie on the line AB?
(2) Does the point $(2, -6)$ lie on the line AB?

What is the sign of the gradient of AB?
Note how the line AB slopes compared with the lines in the last diagram.

IV. Similarly, by considering the co-ordinates of any point on the line CD, and finding its gradient, find the equation of the line CD.

From the above exercises it is clear that the equation $y = mx$ (where m is a constant) has for its graph a straight line, with gradient m, passing through the origin.

If m is $+$, the line slopes upwards from left to right ╱

If m is $-$, the line slopes downwards from left to right ╲

E.g. $2y = 3x$, i.e. $y = \frac{3}{2}x$ has for its graph a straight line through the origin, sloping upwards from left to right, with gradient $\frac{3}{2}$; and $3y + 4x = 0$, i.e. $y = -\frac{4}{3}x$ has for its graph a straight line through the origin, sloping downwards from left to right, with gradient $-\frac{4}{3}$.

Note: The dotted line on the diagram is parallel to the y axis and 3 units from it. The x co-ordinate of every point on this line is 3. The equation of this line is therefore $x = 3$.

What line would have $y = -3$ for its equation?

On the diagram on p. 205 the graph of $y = 2x$ has been drawn as a dotted line. We know that its gradient is 2. The straight line AB has been drawn parallel to the dotted line. Hence its gradient is also 2.

The y co-ordinate of every point on the dotted line is twice its x co-ordinate.

For any value of x, the y co-ordinate of the point on the line AB is 3 more than the y co-ordinate of the point on the dotted line.

Hence the y co-ordinate of every point on the line AB is 3 more than twice its x co-ordinate.

Therefore the equation of AB is $y = 2x + 3$.

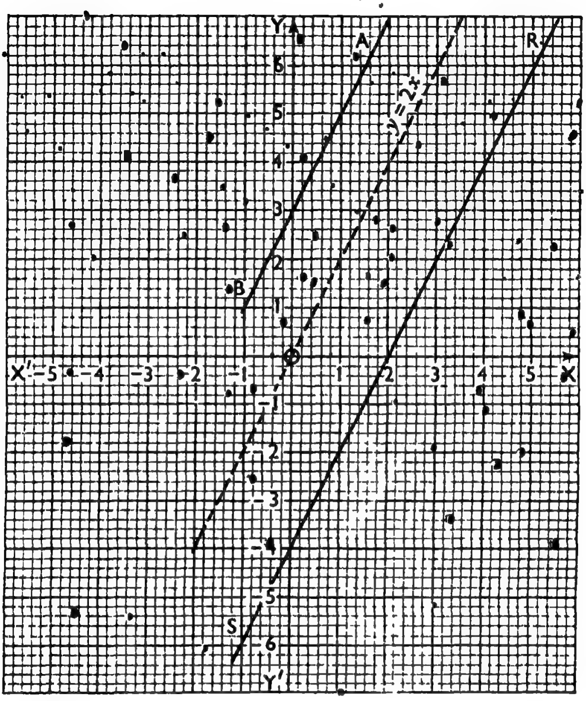
Note that when $x = 0$, $y = 0$ on the dotted line;

when $x = 0$, $y = +3$ on the continuous line.

Therefore the line AB cuts the y-axis at the point $(0, +3)$.

The equation of the line, namely $y = 2x + 3$, tells us at once the gradient of the line, namely 2 (the coefficient of x), and the point where it cuts the y-axis, namely $(0, +3)$ from the constant term $+3$.

Similarly, for any value of x, the y co-ordinate of every point on the line RS is 4 less than that on the dotted line, and



so the equation of RS is $y = 2x - 4$. It cuts the y-axis at the point $(0, -4)$.

We see that the equation $y = 2x + k$, where k is any constant, has for its graph a line parallel to $y = 2x$, so that the equation $y = 2x + k$ represents a whole family of parallel straight lines (when $k = 0$, $y = 2x + k$ becomes $y = 2x$).

$y = 2x + 3$ and $y = 2x - 4$ are both equations of the first degree in x and y. Every equation of the first degree in x and y can be put into the form $y = ax + b$

$$\text{E.g. } 2x - 3y + 5 = 0$$
$$\therefore\ 2x + 5 = 3y$$
$$\therefore\ y = \frac{2x+5}{3} = \tfrac{2}{3}x + \tfrac{5}{3}.$$

This equation represents a straight line, gradient $\frac{2}{3}$, cutting the y axis at the point $(0, +\frac{5}{3})$.

The straight-line graph of the equation $2x - 3y + 5 = 0$ could be drawn from this information. It could also be drawn by finding the co-ordinates of 3 points which must lie on the graph.

$$y = \frac{2x+5}{3}.$$

A table is drawn up showing the values of y for 3 values of x:

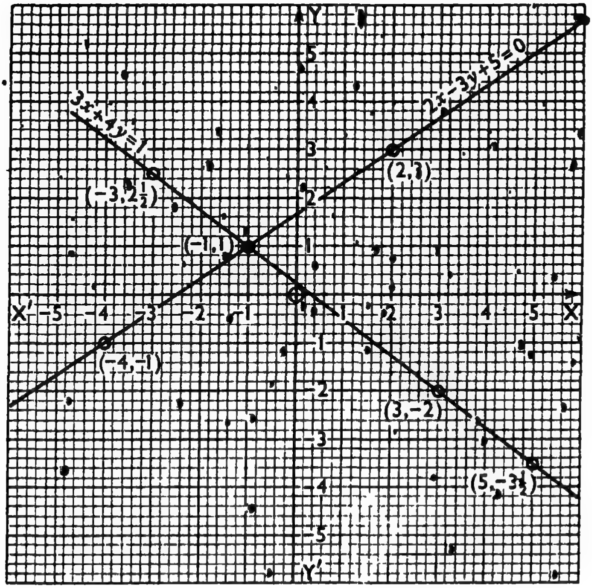
x	-1	2	-4
y	1	3	-1

The points $(-1, 1)$, $(2, 3)$, $(-4, -1)$ are on the graph, and the straight line is drawn through these three points. Two points would have been sufficient to fix the straight line—the third point is added to check the accuracy of the work. The values of x, chosen for drawing up the table, should be such that the points are well apart.

The graph of $2x - 3y + 5 = 0$ is drawn on the diagram on p. 207.

If on the same diagram we draw the straight-line graph of the equation $3x + 4y = 1$,

$$\text{i.e. } y = \frac{1-3x}{4}$$



from the following table of values:

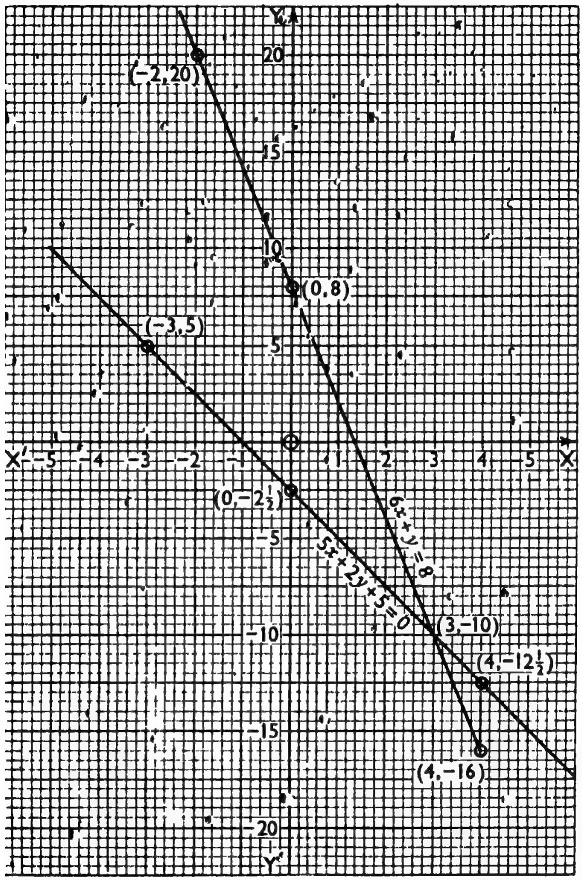
x	-3	3	5
y	$2\frac{1}{2}$	-2	$-3\frac{1}{2}$

we see that the two straight lines cut at the point $(-1, 1)$ and at no other point.

The co-ordinates of every point on the first line satisfy the equation $2x - 3y + 5 = 0$, and the co-ordinates of every point on the second line satisfy the equation $3x + 4y = 1$. The lines intersect in one point, the point $(-1, 1)$.

Hence the co-ordinates of this point, and of this point only, satisfy both equations.

$\therefore$ $x = -1, y = 1$ is the solution of the two simultaneous equations $2x - 3y + 5 = 0$ and $3x + 4y = 1$.



Hitherto, the scales on both axes have been the same. This need not be the case, as is shown in the following example:

Solve graphically $5x + 2y + 5 = 0$, $6x + y = 8$

$$5x + 2y + 5 = 0$$
$$y = -\frac{5x + 5}{2}$$

x	4	0	−3
y	$-12\frac{1}{2}$	$-2\frac{1}{2}$	5

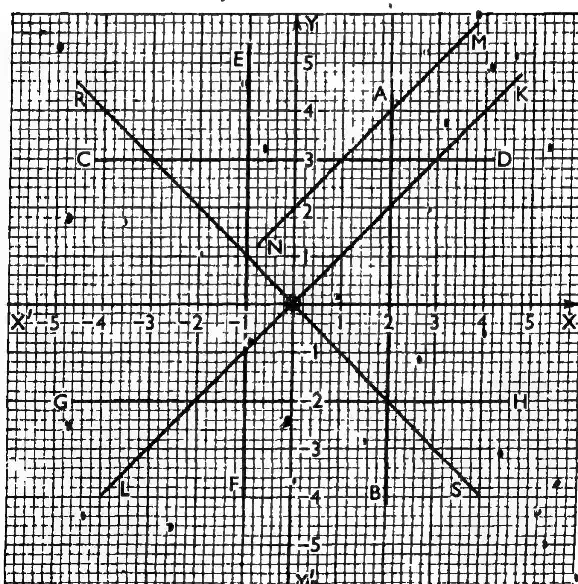
$$6x + y = 8$$
$$y = 8 - 6x$$

x	4	0	−2
y	−16	8	20

The graphs intersect in the point $(3, -10)$.

Solution is $x = 3$,
$y = -10$

Exercises 66



1. On the above diagram, what is the equation of the x-axis, the y-axis?

What is the equation of each of the lines AB, CD, EF, GH, KL, MN, RS?

2. On the same diagram draw the graphs of the following equations:

(i) $y = x + 3$ (iv) $y = 3x - 2$
(ii) $y = x - 2$ (v) $y + x = 2$
(iii) $y = 2x + 1$ (vi) $y + 2x + 3 = 0$

3. Draw the graphs of the following equations:

(i) $3x + y = 4$ (iv) $x + 4 = 0$
(ii) $3x - 2y = 5$ (v) $2y - 3 = 0$
(iii) $2x + 3y = 6$ (vi) $3y + 4x = 7$

4. Solve graphically the following equations:

(i) $2x + y = 4$, $x = 2y - 3$ (v) $2y + 4x + 1 = 0$, $3x + y = -2$
(ii) $3x - 2y = 0$, $x - y = 1$ (vi) $3x + 2y = 2$, $2x - 3y - 10 = 0$
(iii) $2x + 3y = 1$, $y - 3x = 4$ (vii) $x + 2y = 3$, $3x = 6 - 4y$
(iv) $5x - y = 2$, $3y - 4x = 5$ (viii) $3x - 4y = 0$, $3x = 2y - 1$
(ix) $3x + 2y = -11 = 2x + 3y - 7$

5. Draw the graph of $y = x - 1$ and deduce the graph of $y = x - 3$ and of $2y = 2x - 3$.

6. Draw the graph of $y = 3x + 5$ and deduce the graph of $y = 3x + 2$ and of $2y - 6x + 1 = 0$.

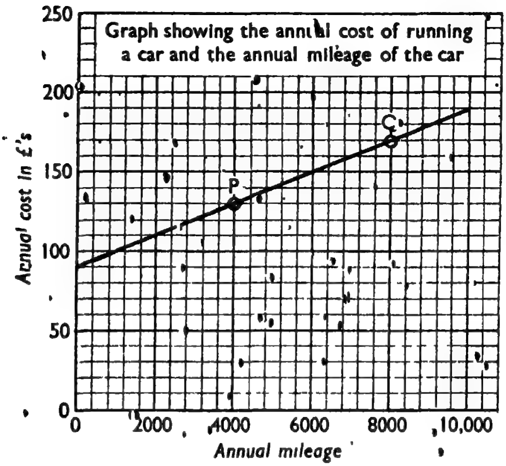
The graph on p. 211 is a straight line.

Let y = annual cost in pounds
x = annual mileage

Since this is a straight line, its equation is of the form $y = ax + b$ when a and b are constants.

Take two points P and Q on the line.

P is the point (4000, 130).



∴ Substituting these values for x and y in the equation we get

$$130 = 4000a + b \quad \ldots \ldots \quad (1)$$

Q is the point (8000, 170).

$$\therefore\ 170 = 8000a + b \quad \ldots \ldots \quad (2)$$

Subtracting equation (1) from equation (2)

$$\therefore\ 40 = 4000a$$

$$\therefore\ a = \frac{1}{100}$$

Then $\quad b = 130 - 4000 \times \frac{1}{100}$ from equation (1)

$$= 90$$

∴ The equation of the line is $y = \frac{1}{100}x + 90$.

This is the law governing the cost of running this car.

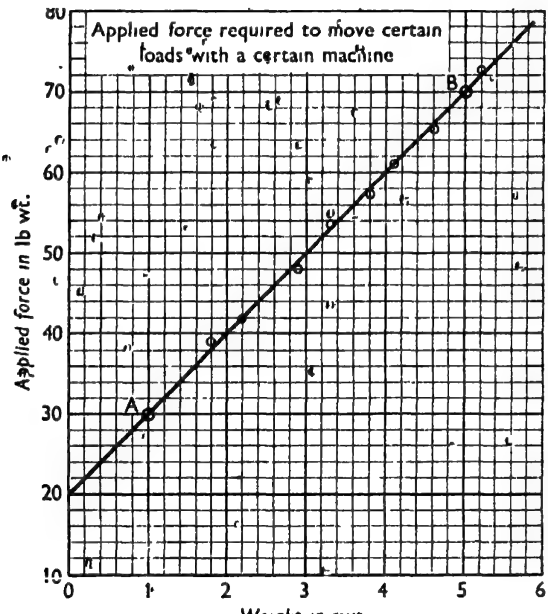
$$\text{Cost (in pounds)} = 90 + \frac{1}{100}x$$

(where x is the number of miles covered per annum).

It was found by experiment with a certain machine that the force (F) in lb. wt. required to move certain loads (W) in cwt. was as follows:

F (lb wt.)	39.0	42.0	48.0	53.4	57.5	61.0	65.5	72.6
W (cwt.)	1.8	2.2	2.9	3.3	3.8	4.1	4.6	5.2

Assuming that the law connecting F and W is a linear one, find it.



The points are plotted as in the diagram above. It will be seen that these points lie approximately in a straight line. The straight line is drawn to pass through them as evenly as possible. Then two points on this line are chosen fairly far apart, e.g. A and B, and the procedure in finding the law is the same as before.

Let the law be $F = aW + b$ (where a and b are constants).

The two points give $F = 30,\ W = 1$

$F = 70,\ W = 5$

Substituting these values in the equation and solving for a and b, we get $a = 10$, $b = 20$.

$\therefore$ The law is $F = 10W + 20$.

Exercises 67

1. What is the equation of the straight lines joining each pair of points?

(i) (0, 4) and (1, 5) (iv) (−1, −5) and (3, 7)

(ii) (1, −2) and (6, 3) (v) (3, −4) and (−3, −4)

(iii) (2, 5) and (−2, −3) (vi) (2, 1) and (−2, −5).

2. In a system of pulleys the forces required to move certain loads were found to be as follows:

Force reqd in lb. wt	1 2	2 3	2 6	3 2	3 8	4 8	5 2
Load in lb	4	8	10	12	15	18	20

Find graphically the law of the system connecting the force (P) and the load (W).

3. In an experiment with a certain mass of gas (volume constant) the pressure of the gas at various temperatures was found to be as follows:

Temp. (t) in ° C.	30°	60	100°	130°	160°	180°	200°
Pressure (P) (in. of mercury)	32 8	36 4	40 0	43 8	46 8	49 3	51 2

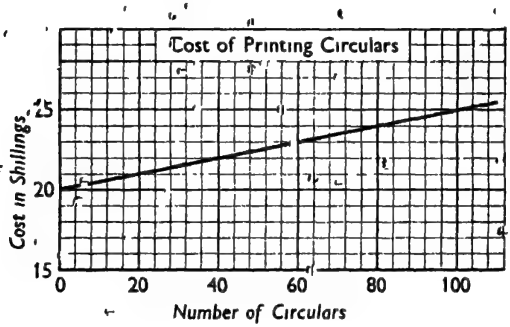
Find graphically the law connecting P and t.

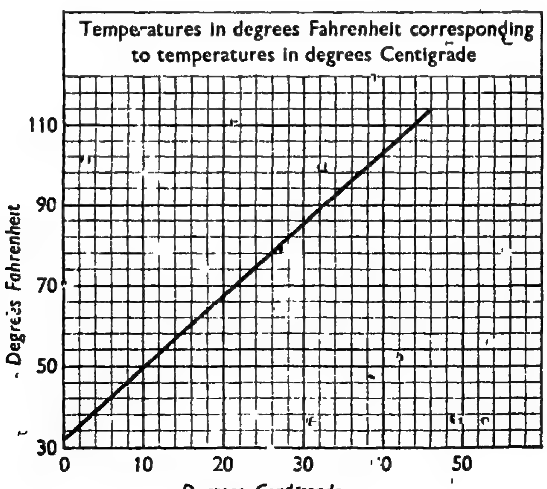
4. The graphs below show:

(*a*) The cost (C) of printing copies of a certain circular when various numbers (N) are printed.

(*b*) Temperature readings in degrees Fahrenheit (F) and corresponding readings in degrees Centigrade (C).

In each case find the law connecting the two quantities.





5. The following table gives the velocity of a body in feet per sec. at certain times after its motion was changed. From a graph find the law connecting the velocity (V) and the time (t).

t (sec.)	1	3	5	8	10	13
Velocity (V) (ft. per sec.)	46·8	40·5	33·8	24·6	18·0	8·2

What was the velocity when the motion was changed? After how many seconds will the body come to rest?

6. A salesman's weekly wages consists of a fixed amount together with a sum dependent on his weekly sales. The following table shows his total wages and his sales in five successive weeks. Find the formula connecting his total wage (W) and his total sales (S).

Total wages (W)	£17 4*s.*	£21 5*s.*	£13 7*s.*	£32 8*s.*	£28 5*s.*
Total sales (S)	£2440	£3250	£1670	£5480	£4650

Paper 26

1. Factorise:

(i) $(x - y)^2 - 4y^2$
(ii) $x^3 + 7x^2 - 8x$
(iii) $x^2 - 6y + 2x - 3xy$

2. Find the quotient and remainder when $6x^3 + 3x^2 - 7$ is divided by $2x - 3$. What must be added to $6x^3 + 3x^2 - 7$ so that the resulting expression has $2x - 3$ as one of its factors? What are then its other factors?

3. (1) A man is allowed a discount of b pence in the shilling. How much does he actually pay for goods marked at £a?

(2) p lb. of China tea are mixed with q lb. of Indian tea. How many ounces of Indian tea are there in $\frac{1}{4}$ lb. of the mixture?

(3) The length of a rectangle is l in. and its area x sq. in. Its length is increased by 50% and its breadth is decreased by 50%. What are its new dimensions? By what fraction has its area been changed, and has its area been increased or decreased?

4. Solve:

(i) $3x^2 = 75$
(ii) $3x^2 + 4x = 4$

5. Solve:

$$\frac{x}{3} + \frac{y}{2} = 1\frac{1}{6}$$
$$x^2 + 2xy - y^2 = 7.$$

6. The area of a certain rectangle is 72 sq. in. If its length were decreased by 4 in. and its breadth increased by 8 in., its area would be increased by $\frac{2}{3}$ sq. ft. Find its length and breadth.

Paper 27

1. Factorise:

(i) $a^2 - 4(1 - 2a)^2$
(ii) $5x^2 - 6xy + 8y^2$
(iii) $xy + 3ay + y^2 - 3ax$

2. If $a = b + 2$ and $c = b - 2$, find the value of:

(i) $2a^2 - b(b + 4) - (c + 4)^2$
(ii) $ab - bc + ca - b(b + 4)$

3. Solve:

(i) $\dfrac{x + 2}{3x - 2} = \dfrac{x}{2x - 2}$
(ii) $\dfrac{2}{x - 2} + \dfrac{3}{x + 3} = \dfrac{5}{x + 4}$

4. Simplify:

(i) $\dfrac{a^2 - 7a + 12}{a^2 - 9} \times \dfrac{a^2 + 2a - 3}{a^2 - 8a + 16}$
(ii) $\dfrac{3x}{3x - 9} - \dfrac{x + 6}{x^2 - 3x} + \dfrac{1}{x}$

5. (i) If x cwt. of sugar cost £x, and the sugar is sold at $\dfrac{x}{2}$ pence per lb., find the least value of x as an integer for a profit to be made.

(ii) A wire forms the boundary of a rectangle, length $2x$ in., breadth x in. The wire is now made to be the boundary of a square. By how much does the area of the square exceed that of the rectangle?

6. Find three positive consecutive even numbers such that the square of the middle one is 408 less than the sum of the squares of the other two.

Paper 28

1. Factorise:

(i) $4x^2 - (2a - 3b)^2$

(ii) $6a^2y + 3ay - 3a$

(iii) $x^3 - 9x^2 + 26x - 24$, given that $x - 2$ is a factor.

2. Simplify:

(i) $\dfrac{a(5a - 9b) + 2b(a - 3b)}{2b(4a - 5b) - 3a(3b - a)}$

(ii) $\left(x - \dfrac{6}{x} - 1\right) \times \left(x + \dfrac{9}{x} + 6\right) \div \left(x^2 + \dfrac{36}{x^2} - 13\right)$

(iii) $(a + b - c)^2 + (b + c - a)^2 - (c + a - b)^2$

3. Solve:

$$\frac{x + y}{3} - \frac{x - y}{2} + 4 = 0$$

$$\frac{x + y}{4} + \frac{x - y}{3} = 5\tfrac{1}{2}$$

4. Simplify:

(i) $\dfrac{2}{x - 1} - \dfrac{3}{1 - x^2} + \dfrac{5}{x + 1}$

(ii) $\dfrac{3}{x^2 + x - 2} + \dfrac{1}{x^2 + 5x + 6} - \dfrac{2}{x^2 + 2x - 3}$

5. Find what value k must have if 5 is one root of the equation $(x + k)^2 + 2(x + k) = 15$.

6. A motorist saved 16 mins on a journey of 48 miles by increasing his average speed by 6 m.p.h. Find his increased speed in m.p.h.

Paper 29

1. Factorise:

(i) $16x^3 - 24x^2y + 9xy^2$

(ii) $2a(b+3) - (3a^2 + 4b)$

(iii) $2x^2 - xy - y^2 - 2x + 2y$

2. (i) If $x = at^2$ and $y = 2at$, find an expression connecting x and y which does not contain t.

(ii) If $x + \frac{1}{x} = a$, and $x - \frac{1}{x} = b$, prove that $x^2 + \frac{1}{x^2} = a^2 - 2$ and that $a^2 = b^2 + 4$.

3. Simplify:

(i) $\left(x - \frac{3}{x+2}\right) \div \left(x + 1 + \frac{4x}{x-3}\right)$

(ii) $\left(1 - \frac{1}{a}\right)\left(1 + \frac{1}{a}\right)\left(a + \frac{1}{a}\right) \div \left(a^2 - \frac{2}{a^2} + 1\right)$

(iii) $(2x + y)^3 - (x - 2y)^3$

4. Solve:

(i) $(5x + 1)(3x - 5) = 3x^2 + 15$

(ii) $\frac{3}{x+5} - \frac{1}{2x+3} = \frac{3}{10}$

5. If $y = \frac{3x+2}{5-4x}$, prove that $x = \frac{5y-2}{4y+3}$. Hence write down the value of x if $y = \frac{ax+b}{c-dx}$.

6. A man bought a second-hand television set for £x. Later he sold it for £25 at a loss of x% of what it cost him. Find x.

Paper 30

1. Factorise:

   (i) $9ab^2 + 6abc - 6db - 4cd$
   (ii) $9x^3 - 9x^2y - 4xy^2 + 4y^3$
   (iii) $3x^2 - 3y^2 - 6x + 3$

2. One factor of $2x^2 - x - 6$ is also a factor of

$$2x^3 - 17x^2 + 12x + 63.$$

Find the factors of both expressions.

3. Solve $x + \frac{3}{4} = \frac{1}{x} + \frac{4}{3}$.

Hence solve $\frac{y+3}{y} + \frac{3}{4} = \frac{y}{y+3} + \frac{4}{3}$.

4. The following table gives the volumes of a certain mass of gas at certain temperatures (pressure being kept constant):

Temperature (°C)	50	110	145	190	250	295
Volume (c.c)	82	96	106	119	131	144

Draw a graph, and find the law connecting the volume (V) and the temperature (t).

5. Simplify:

   (i) $\frac{a+2b}{3a^2 - 5ab - 2b^2} - \frac{a+b}{a^2 - 4b^2}$

   (ii) $\frac{x-1}{x^2+5x+6} + \frac{4(2x+1)}{3x^2+8x+4} - \frac{2x-1}{3x^2+11x+6}$

6. A man bought a certain number of copies of a novel for £8. He sold all but 6 at 3*s*. each above cost price, and the remaining 6 at half the sale price of the others. In all he made a profit of £1 7*s*. How many copies did he buy?

Chapter 22

QUADRATIC EQUATIONS AND PROBLEMS II—COMPLETING THE SQUARE—QUADRATIC SURDS

In Chapter 20 we have considered how to solve Quadratic Equations by the method of factors, and this method should be used whenever possible, but it will be found that there are quadratic equations that cannot be solved by this method, e.g. $2x^2 - 3x - 7 = 0$. They can be solved by a method, called "Completing the Square".

Consider the equation $x^2 = 4$.

If we take the square root of each side we get $\pm x = \pm 2$ (read plus or minus x equals plus or minus 2).

From this equation it would seem that we get four equations, namely:

$$+x = +2$$
$$+x = -2$$
$$-x = +2$$
$$-x = -2$$

Of these four, however,

$$+x = +2, \text{ and } -x = -2$$

give only *one* solution, $x = 2$;

$$\text{and } +x = -2, \text{ and } -x = +2$$

give only *one* solution, $x = -2$

$$\therefore\ x = +2 \text{ or } -2$$

usually written $x = \pm 2$.

Hence when we have an equation in the form $x^2 = 4$ it is sufficient to take the positive square root of x^2, provided we take the positive and negative square root of 4.

Example 1: Solve $(x - 3)^2 = 16$.

Take the square root of each side, remembering that we need take only the positive square root of the left-hand side.

$$\therefore (x - 3) = \pm 4$$
$$\therefore \text{ either } x - 3 = +4, \text{ i.e. } x = 7$$
$$\text{or } x - 3 = -4, \text{ i.e. } x = -1$$
$$\therefore x = 7 \text{ or } -1$$

Note: This equation could have been solved by factorisation

$$(x - 3)^2 - 16 = 0$$
$$[(x - 3) + 4][(x - 3) - 4] = 0$$
i.e.
$$(x + 1)(x - 7) = 0$$
$$\therefore \text{ either } x + 1 = 0 \text{ i.e. } x = -1$$
$$\text{or } x - 7 = 0 \text{ i.e. } x = 7$$

Example 2: "*Completing the Square*".

Solve $3x^2 - 5x + 2 = 0$.

1. Divide both sides by 3 to make the coefficient of x^2 unity.

$$\therefore x^2 - \tfrac{5}{3}x + \tfrac{2}{3} = 0$$

2. Transpose the constant term to the right-hand side.

$$\therefore x^2 - \tfrac{5}{3}x = -\tfrac{2}{3}$$

3. Complete the square on the left-hand side, by adding to both sides of half the coefficient of x.

$$\therefore x^2 - \tfrac{5}{3}x + (\tfrac{5}{6})^2 = -\tfrac{2}{3} + (\tfrac{5}{6})^2$$
$$\therefore (x - \tfrac{5}{6})^2 = -\tfrac{2}{3} + \tfrac{25}{36}$$
$$= \frac{-24 + 25}{36}$$
$$= \tfrac{1}{36}$$

4. Take the square root of each side.

$$\therefore\ x - \tfrac{5}{6} = \pm\tfrac{1}{6}$$

$$\therefore \text{ either } x - \tfrac{5}{6} = +\tfrac{1}{6}, \text{ i.e. } x = \tfrac{5}{6} + \tfrac{1}{6} = 1$$

$$\text{or } x - \tfrac{5}{6} = -\tfrac{1}{6}, \text{ i.e. } x = \tfrac{5}{6} - \tfrac{1}{6} = \tfrac{4}{6} \text{ or } \tfrac{2}{3}$$

$$\therefore\ x = 1 \text{ or } \tfrac{2}{3}$$

Note: This example could have been solved by the factor method, the preferable method in this case.

$$3x^2 - 5x + 2 = 0$$

$$\therefore\ (3x - 2)(x - 1) = 0$$

$$\therefore \text{ either } 3x - 2 = 0 \text{ i.e. } x = \tfrac{2}{3}$$

$$\text{or } x - 1 = 0 \text{ i.e. } x = 1$$

$$\therefore\ x = \tfrac{2}{3} \text{ or } 1$$

Example 3: Solve, correct to two decimal places

$$2x^2 + 9x + 3 = 0$$

$$\therefore\ x^2 + \tfrac{9}{2}x + (\tfrac{9}{4})^2 = -\tfrac{3}{2} + (\tfrac{9}{4})^2$$

$$\therefore\ (x + \tfrac{9}{4})^2 = -\tfrac{3}{2} + \tfrac{81}{16}$$

$$= \frac{-24 + 81}{16}$$

$$= \frac{57}{16}$$

$$\therefore\ x + \tfrac{9}{4} = \pm\frac{\sqrt{57}}{4} = \pm\frac{7{\cdot}550\ldots}{4}$$

$$\therefore \text{ either } x = -\tfrac{9}{4} + \frac{7{\cdot}550\ldots}{4} = \frac{-1{\cdot}450}{4} = -0{\cdot}36$$

$$\text{or } x = -\tfrac{9}{4} - \frac{7{\cdot}550\ldots}{4} = \frac{-16{\cdot}550}{4} = -4{\cdot}14$$

$$\therefore\ x = -0{\cdot}36 \text{ or } -4{\cdot}14$$

Example 4: Solve $x^2 + 4x + 5 = 0$.

$$\therefore\ x^2 + 4x = -5$$

$$\therefore\ x^2 + 4x + (2)^2 = -5 + (2)^2$$

$$\therefore\ (x + 2)^2 = -1$$

Since the square root of -1 is not a real number, there are no real solutions of this equation.

Exercises 68

Solve:

1. $x^2 = 81$
2. $x^2 = 2\frac{1}{4}$
3. $2x^2 = 3\frac{1}{8}$
4. $x^2 = k^2$
5. $(x - 3)^2 = 9$
6. $(x + 2)^2 = 25$
7. $(2x + 3)^2 = 16$
8. $(3x - 2)^2 = 100$
9. $(x - a)^2 = b^2$
10. $9(x + 2)^2 = 16$
11. $(x + 2)^2 = 4x^2$
12. $(2x - 5)^2 = (x - 4)^2$

What terms must be added to each of the following expressions to make it a complete square? In each case state the resulting expression in the form of a square.

13. $x^2 + 4x$
14. $x^2 - 6x$
15. $x^2 + 3x$
16. $x^2 - 5x$
17. $x^2 + \frac{2}{3}x$
18. $x^2 + \frac{5}{4}x$
19. $x^2 - \frac{1}{3}x$
20. $x^2 + 2ax$
21. $x^2 - 3ax$

Solve, by completing the square. Also solve by factorising.

22. $x^2 + 6x = 16$
23. $x^2 - 4x + 3 = 0$
24. $x^2 + x = 2$
25. $2x^2 - 5x + 2 = 0$
26. $x^2 + 2{\cdot}4x + 0{\cdot}8 = 0$
27. $x^2 - 1{\cdot}1x + 0{\cdot}3 = 0$

Solve, by completing the square, giving the answers correct to two decimal places:

28. $x^2 - 2x = 2$
29. $x^2 - 6x - 1 = 0$
30. $x^2 + 2x = 4$
31. $x^2 + 8x - 3 = 0$
32. $x^2 = 4x + 3$
33. $x(x + 3) = 5$
34. $5x = x^2 - 7$
35. $x^2 = 2(3x - 1)$
36. $(x - 2)(x + 3) = 3$
37. $2x^2 + 8x = 5$
38. $2x^2 - 6x = 3$
39. $x(2x - 3) = 1$
40. $(x + 1)(2x - 1) = 4$

Solve, giving the answers correct to two decimal places. Where there are no real solutions, say so.

41. $3x^2 = 2x + 4$
42. $x(1 - 3x) = 3(1 - 2x)$
43. $2x^2 - 3x + 5 = 0$
44. $x(3x + 10) + 4 = 0$
45. $3x(1 - x) = 2(1 - 3x)$
46. $3x(x + 2) = 2(x - 5)$

47. $4x^2 - 2x = 1$

48. $2x + 9 = 4x^2$

49. $6x^2 + 8x + 2 = 0$

50. $4x^2 = 3(x - 3)$

Solution of Quadratic Equations by Use of a Formula

Consider the quadratic equation

$$ax^2 + bx + c = 0$$

$$\therefore \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\therefore \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$= \frac{-4ac + b^2}{4a^2}$$

Take the square root of both sides.

$$\therefore \quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

either

$$x + \frac{b}{2a} = \frac{+\sqrt{b^2 - 4ac}}{2a} \quad \text{i.e. } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

or $$x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a} \quad \text{i.e. } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The solution is often written in the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Using this formula, we can solve any quadratic equation.

e.g. Solve correct to two decimal places $5x^2 + 6x - 3 = 0$

This equation is of the form $ax^2 + bx + c = 0$ where $a = 5$, $b = 6$, $c = -3$.

Substituting these values in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we get $x = \dfrac{-(+6) \pm \sqrt{(+6)^2 - 4(+5)(-3)}}{2(+5)}$

$$= \frac{-6 \pm \sqrt{36 + 60}}{10}$$

$$= \frac{-6 \pm \sqrt{96}}{10}$$

$$= \frac{-6 \pm 9{\cdot}798}{10}$$

$$= \frac{3{\cdot}798}{10} \text{ or } \frac{-15{\cdot}798}{10}$$

$\therefore\ x = 0{\cdot}38$ or $-1{\cdot}58$ (correct to two decimal places)

Exercises 69

Solve, using the formula, giving answers correct to two decimal places:

1. $x^2 + 4x + 2 = 0$
2. $x^2 + 3x - 2 = 0$
3. $x^2 - 2x - 2 = 0$
4. $x^2 + 4x - 1 = 0$
5. $2x^2 - 7x + 4 = 0$
6. $3x^2 + 5x = 7$
7. $5x^2 = 9x - 3$
8. $(3x - 2)(2x + 4) = 4x^2 - 5$
9. $(4x + 3)(4x - 3) = 3(12x - 7)$
10. $\frac{1}{3}x^2 + \frac{1}{2}x + \frac{1}{12} = 0$
11. $(5 + 2x)(3 - x) = 2$
12. $2(x - 3)(x - 2) + 3(x - 3)(x + 5) = (2x + 3)(x - 4)$
13. $x^2 - 1{\cdot}2x + 0{\cdot}3 = 0$
14. $x^2 + 1{\cdot}1x - 0{\cdot}4 = 0$
15. $\frac{1}{3}x^2 - \frac{1}{5}x - \frac{1}{2} = 0$
16. $x^2 + 0{\cdot}1x - 0{\cdot}05 = 0$
17. $x^2 + 1{\cdot}4x - 0{\cdot}04 = 0$

Harder Equations

Example 1: Solve: $\dfrac{x-2}{2(3x-2)} - \dfrac{x+2}{3(x+3)} = \dfrac{1-x^2}{3x^2+7x-6}$.

$$\frac{x-2}{2(3x-2)} - \frac{x+2}{3(x+3)} = \frac{1-x^2}{(3x-2)(x+3)}$$

Multiply both sides by $6(3x-2)(x+3)$.

$$\therefore\ 3(x-2)(x+3) - 2(3x-2)(x+2) = 6(1-x^2)$$
$$\therefore\ 3x^2 + 3x - 18 - 6x^2 - 8x + 8 = 6 - 6x^2$$
$$\therefore\ 3x^2 - 5x - 16 = 0$$

Solving this equation by the formula

$$x = \frac{5 \pm \sqrt{25 + 192}}{6} = \frac{5 \pm \sqrt{217}}{6}$$

$\therefore\ x = 3{\cdot}29$ or $-1{\cdot}62$ (correct to two decimal places)

Example 2: Solve $\dfrac{2}{4x-3} - \dfrac{1}{3x-1} = \dfrac{1}{x+1} - \dfrac{3}{5x+4}$.

Simplifying each side,

$$\therefore\ \frac{2(3x-1) - (4x-3)}{(4x-3)(3x-1)} = \frac{(5x+4) - 3(x+1)}{(x+1)(5x+4)}$$
$$\therefore\ \frac{2x+1}{(4x-3)(3x-1)} = \frac{2x+1}{(x+1)(5x+4)}$$

This equation is true if $2x + 1 = 0$, *i.e. if* $x = -\frac{1}{2}$. *This is one solution.*

If, however, $2x + 1$ *is not equal to zero, we can divide both sides of the equation by* $2x + 1$.

Then
$$\frac{1}{(4x-3)(3x-1)} = \frac{1}{(x+1)(5x+4)}$$
$$\therefore\ (4x-3)(3x-1) = (x+1)(5x+4)$$
$$\therefore\ 12x^2 - 13x + 3 = 5x^2 + 9x + 4$$
$$\therefore\ 7x^2 - 22x - 1 = 0$$

Solving by the formula $x = 3{\cdot}19$ or $-0{\cdot}04$.

$\therefore\ x = -\frac{1}{2}$ or $3{\cdot}19$ or $-0{\cdot}04$ (correct to two decimal places)

Exercises 70

Solve, giving the answers, where they are not exact, correct to two decimal places. Where there are no real solutions state this.

1. $x + 1 = \dfrac{2}{x}$

2. $x - \dfrac{7}{x} - \dfrac{3}{2} = 0$

3. $\dfrac{2}{5x} + x = \dfrac{8}{5}$

4. $x - 1 = 3\left(\dfrac{2}{x} + 3\right)$

5. $2x - 5 = 2\left(2 - \dfrac{3}{x}\right)$

6. $\dfrac{x + 2}{x} = \dfrac{x}{2x - 5}$

7. $\dfrac{x - 3}{x + 1} = \dfrac{2x - 3}{x + 5}$

8. $\dfrac{3x - 1}{x + 1} = \dfrac{x - 2}{x}$

9. $\dfrac{3x^2 + 7}{3x - 5} = 2x$

10. $\dfrac{4x + 3}{3x + 5} - \dfrac{2x + 5}{x + 2} = 0$

11. $\dfrac{2}{x - 1} - 3(x - 2) = 3$

12. $\dfrac{1}{x + 1} - \dfrac{1}{x} = \dfrac{2}{x - 1}$

13. $\dfrac{1}{x + 3} - \dfrac{2}{x - 2} + \dfrac{3}{x - 1} = 0$

14. $\dfrac{2}{x} + \dfrac{3}{x - 3} = \dfrac{1}{x + 2}$

15. $\dfrac{3}{x - 6} = \dfrac{1}{3} + \dfrac{3}{x - 4}$

16. $\dfrac{x + 1}{3x - 2} - \dfrac{1}{x} = \dfrac{2}{3}$

17. $\dfrac{3}{x + 5} - \dfrac{1}{2x + 7} = \dfrac{1}{3}$

18. $\dfrac{2x - 5}{x - 2} - \dfrac{7}{3x^2 - 4x - 4} = \dfrac{x + 1}{3x + 2}$

19. $\dfrac{2x - 3}{x + 3} - \dfrac{x - 3}{2x + 3} = 1$

20. $\dfrac{x + 1}{2x + 1} + \dfrac{2x - 1}{x - 1} = 2$

21. $\dfrac{6}{x - 2} - \dfrac{1}{(x - 2)^2} = 2$

22. $\dfrac{6}{(3x + 2)^2} = 1 + \dfrac{1}{3x + 2}$

23. $(x - 1)^2 = \left(\dfrac{5}{x + 1}\right)^2$

24. $\dfrac{1}{3}\left(\dfrac{5}{x} - 3\right) = \dfrac{1}{2}\left(\dfrac{2}{x + 1} + \dfrac{5}{x}\right)$

25. $\frac{3}{2x-1} - \frac{1}{x-2} = \frac{2}{x-1} - \frac{1}{x-3}$

26. $\frac{3}{2x+1} - \frac{2}{x+1} = \frac{4}{3x+1} - \frac{3}{2x+1}$

27. $\frac{1}{x-1} - \frac{1}{2x-1} = \frac{2}{x+2} - \frac{1}{x+1}$

28. $\frac{2}{3x-4} - \frac{1}{2x+3} = \frac{1}{x-2} - \frac{4}{5x+2}$

29. $\frac{2}{2x-1} - \frac{3}{5x-1} = \frac{1}{x+2} - \frac{2}{6x+5}$

30. $\frac{x-1}{x-2} - \frac{x+2}{x+1} = \frac{x}{x-1} - \frac{x+4}{x+3}$

Problems

Example 1: A man bought a certain number of articles for £3. If each article had cost 1s less, he would have obtained 16 more for the same money. How many articles did he buy?

Let x = number of articles bought.

$\therefore$ Cost of each $= \frac{60}{x}$ shillings.

If 16 more had been obtained the number of articles would have been $x + 16$

Cost of each now $= \frac{60}{x+16}$ shillings.

But this is 1 shilling less than the previous cost.

$$\therefore \frac{60}{x} - \frac{60}{x+16} = 1$$

$$\therefore 60(x+16) - 60x = x^2 + 16x$$

$$\therefore x^2 + 16x - 960 = 0$$

$$\therefore (x+40)(x-24) = 0$$

$$\therefore x = 24 \text{ or } x = -40$$

$$\therefore \text{Number of articles} = 24$$

Note: The method above is the one that would generally be used, but this problem could have been solved by letting x shillings be the cost of one article. The student should work the

problem thus, and realise that in solving problems there are often various methods of approach. The next problem illustrates this.

Example 2: The substitution of a diesel train for a steam train reduces the time for a journey of 48 miles by 36 min. If the average speed of the two trains differs by 18 m.p.h., find the average speed of the diesel train.

Let x hr. = time taken for the journey by the diesel train.

$\therefore$ Average speed of diesel train $= \frac{48}{x}$ m.p.h.

$(x + \frac{3}{5}$ hr.) = time taken for the journey by the steam train.

$\therefore$ Average speed of steam train $= \frac{48}{x + \frac{3}{5}}$ m.p.h.

$$\therefore \frac{48}{x} - \frac{48}{x + \frac{3}{5}} = 18$$
$$\therefore 48(x + \tfrac{3}{5}) - 48x = 18x(x + \tfrac{3}{5})$$
$$\therefore 18x(x + \tfrac{3}{5}) = 48 . \tfrac{3}{5}$$
$$\therefore 18x(5x + 3) = 144$$
$$\therefore x(5x + 3) = 8$$
$$\therefore 5x^2 + 3x - 8 = 0$$
$$(5x + 8)(x - 1) = 0$$
$$\therefore x = 1 \text{ or } -\tfrac{8}{5}$$

$\therefore$ Time taken by diesel train = 1 hr.

$\therefore$ Average speed of diesel train = 48 m.p.h.

Exercises 71

1. The sum of a number and 9 times its reciprocal is 6. Find the number.

2. A number exceeds 12 times its reciprocal by 4. Find the number.

3. The sum of the reciprocals of two positive whole numbers is $\frac{7}{24}$. If the numbers differ by 2, find the numbers.

4. Some boys spent 5*s*. in buying ice-cream cones. They would have got 10 more if each cone had cost 1*d*. less. What was the price of a cone, and how many did they buy?

5. An increase in the price of a notebook by $9d.$ means that 6 fewer are got for £1. How many did £1 buy before the increase?

6. When the price of eggs falls by $1s.$ per dozen, 2 dozen more can be got for £2. Find the higher price of the eggs per dozen.

7. A school class plans an outing to cost £8. At the last minute 8 pupils are prevented by illness from taking part, and the remaining pupils agree to pay an extra $1s.$ each, and refund the original subscription to the absentees. How many pupils are in the class?

8. For a certain excursion an adult's fare is $2s.$ more than a child's. The total fares amounted to £9 $6s.$, £7 $10s.$ of this being the amount of the adult fares. 42 persons took part in the excursion. How many children took part?

9. A club has Town Members and Country Members, each of the former paying £4 more per year for membership. The total subscriptions are as follows: Town Members £2100, Country Members £600. There are 500 members altogether. What is the annual subscription of a Country Member?

10. A man bought a certain number of articles for £150. On delivery he found that 10 were so badly damaged as to be unsaleable, and calculated that, if he sold the others at $15s.$ more each than their cost price, he would just make altogether what he paid for all the articles. How many articles did he buy?

11. A woman buys a television set for £70. She pays half of it and agrees to pay the remainder and also £1 as interest, by weekly instalments. She finds that if she increases her weekly payments by $6s.$ she can complete her payments 10 weeks earlier. Find her weekly payment if she chooses to pay the extra $6s.$ each week.

12. An increase of speed by 6 m.p.h. on a journey of 90 miles reduces the time taken for the journey by $\frac{1}{2}$ hr. Find the increased speed.

13. A decrease of speed by 10 m.p.h. on a journey of 80 miles increases the time taken for the journey by 40 min. Find the decreased speed.

14. Due to a fault developing on a journey when 81 miles had still to be done, an express train had to reduce its speed by 18 m.p.h. It arrived at 3.15 p.m. instead of at 2.30 p.m. Find when the fault developed.

15. A man wishes to go to a place $2\frac{1}{4}$ miles distant, and finds that he can save 36 min. by taking the bus both ways, instead of taking the bus there and walking back. The bus travels 12 m.p.h. faster than the man can walk. Find the average speed of the bus.

16. The distance from Glasgow to London is 400 miles by rail and 392 miles by road. A car whose average speed is 32 m.p.h. less than that of an express train takes $7\frac{1}{2}$ hr. longer for the journey. Find the speed of the express train.

17. One man takes 3 days longer than another to do a certain piece of work. Working together, they can do the same work in 2 days. How long does each take when working alone?

18. One pipe takes 3 min. less time than another to fill a certain tank. If both are used the tank can be filled in $6\frac{2}{3}$ min. How long does each pipe take to fill the tank?

19. A certain positive fraction has its numerator 2 less than its denominator. If the numerator and denominator are each decreased by 1, the value of the fraction is decreased by $\frac{1}{21}$. Find the fraction.

20. A square and a rectangle have the same perimeter. The rectangle is 6 in. broad, and its area is 16 sq. in. less than that of the square. Find the area of the square.

Quadratic Surds

When the equation $x^2 + x - 1 = 0$ is solved, we get

$$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

It is impossible to find the exact value of the roots, because it is impossible to find the exact value of $\sqrt{5}$. (We can, as we have seen with similar examples, find the roots correct to any number of places of decimals.) There is no number, which, when squared, gives exactly 5 for the answer. $\sqrt{5}$ is called a *quadratic surd*

If the square root of any number cannot be found exactly, then that square root is called *a quadratic surd*.

$\sqrt{5}$ is taken to mean the positive square root of 5.

$\sqrt{4}$ is not a quadratic surd, since its value is 2.

$\sqrt{4}$ is merely written in surd form.

In a later chapter it is proved that

$$\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$$

and similarly that

$$\sqrt{10} = \sqrt{5 \times 2} = \sqrt{5} \times \sqrt{2}$$
$$\text{Also } \sqrt{2} \times \sqrt{2} = 2$$

Hence $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2 \times \sqrt{3} = 2\sqrt{3}$
and $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5 \times \sqrt{2} = 5\sqrt{2}$

The steps above can be reversed, e.g.

$$2\sqrt{3} = \sqrt{4} \times \sqrt{3} = \sqrt{4 \times 3} = \sqrt{12}$$

When $2\sqrt{3}$ is written as $\sqrt{12}$, it is said to be written as an *entire surd*.

Similarly, $3\sqrt{2}$, written as an entire surd, is

$$\sqrt{9} \times \sqrt{2} = \sqrt{18}$$

Addition and Subtraction of Surds

Just as $3x + 2x = 5x$ and $7x - 2x = 5x$

so $3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$ and $7\sqrt{3} - 2\sqrt{3} = 5\sqrt{3}$

An expression such as $2\sqrt{3} + 3\sqrt{2}$ cannot be written down more simply. It could, of course, be evaluated correct to any number of decimal places.

Example 1: Simplify $\sqrt{8} + \sqrt{32} - \sqrt{18}$.

$$\begin{aligned}\sqrt{8} + \sqrt{32} - \sqrt{18} &= \sqrt{4 \times 2} + \sqrt{16 \times 2} - \sqrt{9 \times 2}\\ &= 2\sqrt{2} + 4\sqrt{2} - 3\sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

Example 2: Simplify $5\sqrt{3} + \sqrt{72} - \sqrt{50} - \sqrt{27}$.

$$\begin{aligned}5\sqrt{3} + \sqrt{72} - \sqrt{50} - \sqrt{27}\\ &= 5\sqrt{3} + \sqrt{36 \times 2} - \sqrt{25 \times 2} - \sqrt{9 \times 3}\\ &= 5\sqrt{3} + 6\sqrt{2} - 5\sqrt{2} - 3\sqrt{3}\\ &= 2\sqrt{3} + \sqrt{2}\end{aligned}$$

Multiplication of Surds

Example 1:

$$\begin{aligned}3\sqrt{2} \times 4\sqrt{3} &= 3 \times \sqrt{2} \times 4 \times \sqrt{3}\\ &= 3 \times 4 \times \sqrt{2} \times \sqrt{3} = 12\sqrt{6}\end{aligned}$$

Example 2:

$$\begin{aligned}4\sqrt{3} \times 3\sqrt{6} = 4 \times \sqrt{3} \times 3 \times \sqrt{6} &= 4 \times 3 \times \sqrt{3} \times \sqrt{6}\\ &= 4 \times 3 \times \sqrt{18}\\ &= 4 \times 3 \times \sqrt{9 \times 2}\\ &= 4 \times 3 \times 3\sqrt{2}\\ &= 36\sqrt{2}\end{aligned}$$

Division of Surds

Example 1: Evaluate $6 \div \sqrt{2}$, correct to two decimal places.

This could be done by calculating $\sqrt{2}$ and dividing. It is much easier, however, to proceed as follows:

$$\frac{6}{\sqrt{2}} = \frac{6 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$
$$= 3 \times 1{\cdot}414$$
$$= 4{\cdot}24 \text{ (correct to two decimal places).}$$

Here we multiply the numerator and the denominator of the fraction by the same number $\sqrt{2}$. $\sqrt{2}$ is chosen because; when the multiplication is performed, the denominator is no longer a surd. This process is called *rationalising the denominator.*

Example 2: $9\sqrt{2} \div 2\sqrt{6}$.

$$\frac{9\sqrt{2}}{2\sqrt{6}} = \frac{9\sqrt{2} \times \sqrt{6}}{2\sqrt{6} \times \sqrt{6}} = \frac{9\sqrt{12}}{2 \times 6} = \frac{9 \times 2\sqrt{3}}{2 \times 6} = \frac{3\sqrt{3}}{2}$$

Example 3: Simplify $\sqrt{27} + \frac{6}{\sqrt{3}} - \sqrt{12}$.

$$\sqrt{27} + \frac{6}{\sqrt{3}} - \sqrt{12} = 3\sqrt{3} + \frac{6\sqrt{3}}{3} - 2\sqrt{3}$$
$$= 3\sqrt{3} + 2\sqrt{3} - 2\sqrt{3}$$
$$= 3\sqrt{3}$$

Example 4: $(\sqrt{3} + 5)^2 = (\sqrt{3})^2 + 2(\sqrt{3})5 + 5^2$
$= 3 + 10\sqrt{3} + 25 = 28 + 10\sqrt{3}$

Exercises 72

Express the following in their simplest forms:

1. $\sqrt{8}$	**4.** $\sqrt{32}$	**7.** $\sqrt{45}$
2. $\sqrt{18}$	**5.** $\sqrt{50}$	**8.** $\sqrt{24}$
3. $\sqrt{27}$	**6.** $\sqrt{48}$	**9.** $\sqrt{80}$

Express the following as entire surds:

10. $2\sqrt{3}$ **13.** $2\sqrt{5}$ **16.** $3\sqrt{10}$

11. $3\sqrt{2}$ **14.** $3\sqrt{6}$ **17.** $2\sqrt{8}$

12. $3\sqrt{3}$ **15.** $5\sqrt{5}$ **18.** $10\sqrt{2}$

Express the following in their simplest forms:

19. $\sqrt{3} \times \sqrt{3}$ **23.** $\sqrt{5} \times \sqrt{10}$ **27.** $\sqrt{6} \times 3\sqrt{2}$

20. $\sqrt{3} \times \sqrt{6}$ **24.** $\sqrt{6} \times \sqrt{12}$ **28.** $2\sqrt{8} \times \sqrt{2}$

21. $\sqrt{2} \times \sqrt{6}$ **25.** $2\sqrt{3} \times 3\sqrt{2}$ **29.** $6\sqrt{3} \quad \sqrt{12}$

22. $\sqrt{2} \times \sqrt{10}$ **26.** $2\sqrt{5} \times 2\sqrt{10}$ **30.** $\sqrt{12} \quad \sqrt{50}$

Evaluate, correct to two decimal places:

31. $\dfrac{1}{\sqrt{2}}$ **32.** $\dfrac{4}{\sqrt{2}}$ **33.** $\dfrac{6}{\sqrt{3}}$

Simplify by rationalising the denominator:

34. $\dfrac{\sqrt{2}}{\sqrt{3}}$ **35.** $\dfrac{3\sqrt{2}}{2\sqrt{3}}$ **36** $\dfrac{2\sqrt{8}}{\sqrt{6}}$

37. $\dfrac{\sqrt{12}}{3\sqrt{6}}$ **38.** $\dfrac{\sqrt{3}}{\sqrt{12}}$ **39.** $\dfrac{3\sqrt{6}}{\sqrt{15}}$

Simplify:

40. $\sqrt{3} + \sqrt{12}$ **42.** $\sqrt{18} + \sqrt{27} - \sqrt{48}$

41. $\sqrt{8} + \sqrt{32}$ **43.** $\sqrt{18} + \sqrt{12} - \sqrt{8}$

44. $\sqrt{50} + \sqrt{108} - \sqrt{2} + \sqrt{27}$

45. $3\sqrt{20} + \sqrt{72} - 3\sqrt{2} - 2\sqrt{45}$

46. $\dfrac{10}{\sqrt{2}} - \sqrt{8}$ **49.** $(2 + \sqrt{3})^2$

50. $(\sqrt{5} - 3)^2$

47. $\sqrt{24} + \dfrac{12}{\sqrt{6}} \quad \dfrac{12\sqrt{2}}{\sqrt{3}}$ **51.** $\sqrt{3}(\sqrt{2} - \sqrt{6})$

48. $\sqrt{5} - \dfrac{10}{\sqrt{5}} + \sqrt{45}$

Solve, leaving the roots in surd form:

52. $x^2 - x - 3 = 0$ **53.** $2x^2 + x - 5 = 0$

CHAPTER 23

LITERAL EQUATIONS—CHANGING THE SUBJECT OF A FORMULA

IN previous chapters letters occurring in equations have generally represented unknown numbers.

We now consider equations, already referred to on page 82, in which the coefficients of the unknowns and the constant terms consist of letters, usually taken from the beginning of the alphabet (a, b, c, etc.). The unknown numbers are usually represented by letters, such as x, y, z, etc., taken from the end of the alphabet.

These equations are called Literal Equations.

Equations of the First Degree with One Unknown

Example 1: Solve the equation

$$\frac{x+a}{b}+\frac{x+b}{a}+2=0$$

Multiply both sides of the equation by ab, the L.C.M. of the denominators.

$$\therefore\ a(x+a)+b(x+b)+2ab=0$$
$$\therefore\ ax+a^2+bx+b^2+2ab=0$$
$$\therefore\ ax+bx=-(a^2+2ab+b^2)$$
$$\therefore\ x(a+b)=-(a+b)^2$$
$$\therefore\ x=-(a+b)$$

Exercises 73

Solve the equations:

1. $2x+a=x+2a$
2. $x+b=2x-c$
3. $2(x-a)=x-b$
4. $ax=a^2-a$

5. $a(x - a) = a - a^2$

6. $ax - b = x - ab$

7. $ax - a^2 = bx - b^2$

8. $ax - a^2b = -bx + b^3$

9. $a(ax - 1) = b(bx - 1)$

10. $a^2bx - a^2 = ab^2x - b^2$

11. $a(x - a) = b(x + a - 2b)$

12. $(a + b)x - 2a^2 = 2ab - (a - b)x$

13. $\dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}$

14. $\dfrac{x}{a} - \dfrac{x}{b} = \dfrac{b}{a} - \dfrac{a}{b}$

15. $\dfrac{a}{x} - \dfrac{b}{x} = \dfrac{1}{b} - \dfrac{1}{a}$

16. $\dfrac{a}{a + x} = \dfrac{b}{b + x}$

17. $\dfrac{x + a}{x - 2a} = \dfrac{a + b}{a - 2b}$

18. $\dfrac{x + a - b}{a + b} = \dfrac{x + a}{a}$

19. $\dfrac{x + 2a}{b} + \dfrac{x + 2b}{a} = \dfrac{2(a + b)^2}{ab} - 3$

20. $\dfrac{(ax - b)(bx - a)}{ab} = (x - 1)^2$

21. $\dfrac{x + a}{a} - \dfrac{x + b}{b} = \dfrac{4x + a + b}{a - b}$

22. $\dfrac{x - a}{2a} + \dfrac{x + b}{2b} = \dfrac{2(x - a + b)}{a + b}$

23. $\dfrac{1}{x + a} + \dfrac{1}{x + b} = \dfrac{2}{x}$

24. $\dfrac{2}{x} - \dfrac{1}{x + 2a} = \dfrac{1}{x - a}$

25. $\dfrac{1}{x + a} + \dfrac{1}{x - 2a} = \dfrac{2}{x - a}$

26. $\dfrac{1}{x + a} + \dfrac{1}{x + b} = \dfrac{2}{x + a + b}$

27. $\dfrac{a}{x + 2a} - \dfrac{b}{x - b} = \dfrac{a - b}{x + a + b}$

28. $\dfrac{b}{ax + b} - \dfrac{a}{bx + a} = \dfrac{b^2 - a^2}{abx}$

Simultaneous equations of the First Degree with Two Unknowns.

Example: Solve for x and y,

$$px + qy = p^2 + pq + q^2, \quad \ldots \quad (1)$$
$$x - y = p \quad \ldots \quad (2)$$

Multiply (2) by q.

$$\therefore qx - qy = pq \quad \ldots \quad (3)$$

Add (1) and (3).

$$\therefore px + qx = p^2 + 2pq + q^2$$
$$\therefore x(p + q) = (p + q)^2$$
$$\therefore x = p + q$$

Substitute in (2).

$$\therefore p + q - y = p$$
$$\therefore y = q$$

$$\boxed{\begin{aligned} x &= p + q \\ y &= q \end{aligned}}$$

Exercises 74

Solve the following equations:

1. $2x + y = 5a$
   $x - y = a$
2. $4x - 3y = 10a$
   $3x + y = a$
3. $\frac{x}{3} + \frac{y}{4} = 2a$
   $\frac{5x}{6} + \frac{3y}{8} = 4a$
4. $3x - 2y = a + 5b$
   $2x + 3y = 5a - b$
5. $\frac{x}{2} + \frac{y}{3} = 2a - b$
   $3x + 4y = 12(a - b)$
6. $2x - 5y = 4a + 5b$
   $bx + 2ay = 0$
7. $x + y = 3(a + b)$
   $ax - by = a^2 - b^2$
8. $\frac{x}{a} + y = a + b$
   $x - by = a^2 - b^2$

9. $ax + by = a^2 + 2ab - b^2$
$bx + ay = a^2 + b^2$

10. $ax + by = (a + b)^2$
$bx - ay = 2b^2$

11. $(a + b)x - (a - b)y = a^2 + b^2$
$ax + by = a^2 + b^2$

12. $\dfrac{x}{a} + \dfrac{y}{b} = 2a$
$x + 2y + b^2 = (a + b)^2$

13. $\dfrac{x}{a} - \dfrac{y}{b} = 0$
$x + y = (a + b)^2$

14. $\dfrac{x}{a} - \dfrac{y}{b} = a - b$
$\dfrac{x}{a - 1} + \dfrac{y}{b + 1} = a + b$

15. $\dfrac{x}{a + b} + \dfrac{y}{a - b} = 2a$
$x - y = 4ab$

16. $\dfrac{a}{x} - \dfrac{b}{y} = a^3 - b^3$
$\dfrac{1}{ax} + \dfrac{1}{by} = a + b$

17. $\dfrac{b}{x} + \dfrac{a}{y} = a^2 + b^2$
$\dfrac{a + b}{x} - \dfrac{a - b}{y} = 4ab$

18. $\dfrac{2}{x} + \dfrac{1}{y} = (a + b)^2$
$\dfrac{a}{x} - \dfrac{b}{y} + b^3 = 0$

19. $\dfrac{1}{x} - \dfrac{1}{y} = a - c$
$\dfrac{c}{x} - \dfrac{a}{y} = b(c - a)$

20. $(a + b)x + (a - b)y = 2a(x + y) = 4ab$

Quadratic Equations

Solve $x^2 - (a + b)x + ab = 0$.

$$x^2 - (a + b)x + ab = 0$$
$$\therefore\ (x - a)(x - b) = 0$$
$$\therefore\ x = a \text{ or } b$$

Exercises 75

Solve:

1. $x^2 + (a - b)x - ab = 0$
2. $x^2 + (a + b)x + ab = 0$
3. $x^2 - (m - n)x - mn$
4. $x^2 + 2ax + a^2 = 0$
5. $x^2 - 2kx + k^2 = 0$
6. $x^2 - a^2 = 0$
7. $x(x + a) = b(x + a)$
8. $\dfrac{a}{x - a} = \dfrac{x - b}{b}$
9. $x^2 - (2a + b)x + 2ab = 0$
10. $x^2 + (5a - 2s)x - 10as = 0$

Changing the Subject of a Formula

The formula $I = \dfrac{PRT}{100}$ enables us to calculate the value in pounds of the Simple Interest, I, on £P invested for T years at R% per annum.

I is called the subject of the formula. When the value of I is known and also the values of R and T, we may calculate the value of P, because the above formula may be written

$$P = \frac{100I}{RT}$$

P is now the subject of the formula, and we have changed the subject of the formula from I to P.

Example 1: Change the subject of the formula $s = \dfrac{a}{1 - r}$ to r

$$s = \frac{a}{1 - r}$$

$$\therefore s(1 - r) = a$$

$$\therefore s - rs = a$$

$$\therefore rs = s - a$$

$$\therefore r = \frac{s - a}{s}$$

Example 2: If $R = \dfrac{r_1 r_2}{r_1 + r_2}$, change the subject to r_1.

$$R = \frac{r_1 r_2}{r_1 + r_2}$$
$$\therefore R(r_1 + r_2) = r_1 r_2$$
$$\therefore Rr_1 + Rr_2 = r_1 r_2$$
$$\therefore Rr_1 - r_1 r_2 = -Rr_2$$
$$\therefore r_1(r_2 - R) = Rr_2$$
$$\therefore r_1 = \frac{Rr_2}{r_2 - R}$$

Example 3: If $T = 2\pi\sqrt{\dfrac{k}{MH}}$, change the subject to H.

$$T = 2\pi\sqrt{\frac{k}{MH}}$$
$$\therefore T^2 = 4\pi^2 \cdot \frac{k}{MH}$$
$$\therefore MHT^2 = 4\pi^2 k$$
$$\therefore H = \frac{4\pi^2 k}{MT^2}$$

Exercises 76

Change the subject of the following formulæ:

1. $V = \frac{1}{3}\pi r^2 h$ to h
2. $A = \pi r^2$ to r
3. $A = \pi r(r + l)$ to l
4. $v^2 = u^2 + 2fs$ to f
5. $s = ut + \frac{1}{2}ft^2$ to f
6. $v = v_0(1 + \alpha t)$ to α
7. $T = 2\pi\sqrt{\dfrac{l}{g}}$ to l
8. $E = \frac{1}{2}mv^2$ to v
9. $s = \dfrac{n}{2}[2a + \overline{n - 1}d]$ to d
10. $T = a + \overline{n - 1}d$ to n
11. $C = \frac{5}{9}(F - 32)$ to F
12. $\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$ to r_1
13. $\dfrac{1}{v} + \dfrac{1}{u} = \dfrac{1}{f}$ to u
14. $A = P\left(1 + \dfrac{r}{100}\right)$ to r
15. $F = \dfrac{mv^2}{r}$ to v
16. $f = \dfrac{m_1 - m_2}{m_1 + m_2}g$ to m_2

17. $c = \dfrac{v_2 c_1}{v_1 - v_2}$ to v_2

18. $I = \dfrac{E}{R + \dfrac{r}{n}}$ to n

19. $I = \dfrac{mnE}{mR + nr}$ to m

20. $H = \dfrac{2\pi RN(W - P)}{33{,}000}$ to P

21. $V = \dfrac{C_1V_1 + C_2V_2}{C_1 + C_2}$ to C_1

22. $V_1 = V_2\left(1 + \dfrac{C_2}{C_1}\right)$ to C_1

23. $\dfrac{\mu}{v} - \dfrac{1}{u} = \dfrac{\mu - 1}{r}$ to u

24. $A = \pi r^2 + \pi r\sqrt{(h^2 + r^2)}$ to h

25. $T = 2\pi\sqrt{\dfrac{l(1 + R^2/r^2)}{g}}$ to r

26. $v = \dfrac{M + m}{m}\sqrt{2gl\left(1 - \dfrac{x}{l}\right)}$ to l

27. $m = \left(\dfrac{V + v}{V}\right)^2 M$ to V

CHAPTER 24

RATIO AND PROPORTION I

Ratio

Two numbers can be compared by finding the ratio that one bears to the other.

If a and b are any two numbers the ratio of a to b is measured by the fraction $\frac{a}{b}$.

The ratio is sometimes written in the form $a : b$.

a and b are called the terms of the ratio.

In the same way we may compare two quantities provided they are of the same kind.

To express the ratio of £a to b half-crowns we reduce the quantities to the same unit. The ratio is then written as

$\frac{8a}{b}$ if both quantities are reduced to half-crowns,

or $\frac{40a}{5b}$ if both quantities are reduced to sixpences,

or $\frac{240a}{30b}$ if both quantities are reduced to pence,

and, of course, $$\frac{240a}{30b} = \frac{40a}{5b} = \frac{8a}{b}$$

Expressed in the form $\frac{8a}{b}$, the ratio is said to be in its lowest terms.

It is impossible to find a ratio between two quantities of different kinds, e.g. between a distance and a weight.

Example 1: Find the ratio between £1 2*s*. 3*d*. and 3*s*. 3*d*.

$$\text{Ratio} = \frac{£1\ 2s.\ 3d.}{3s.\ 3d.} = \frac{89 \text{ threepences}}{13 \text{ threepences}} = \frac{89}{13}$$

Example 2: Find the ratio between the two prices, 8*a* pence per dozen and *a* shillings per score.

$$\text{Ratio} = \frac{8a \text{ pence per dozen}}{a \text{ shillings per score}}$$

$$= \frac{\frac{8a}{12} \text{ pence each}}{\frac{12a}{20} \text{ pence each}}$$

$$= \frac{8a}{12} \times \frac{20}{12a}$$

$$= \frac{10}{9}$$

The equation $x : y = a : b$ means that the ratios $\frac{x}{y}$ and $\frac{a}{b}$ are equal,

$$\text{i.e. } \frac{x}{y} = \frac{a}{b}$$

This gives us no information regarding the actual values of x and y, and in particular we cannot say that $x = a$ and $y = b$.

$$\text{But, since } \frac{x}{y} = \frac{a}{b}$$

$$\therefore\ bx = ay$$

and, dividing throughout by ab,

$$\frac{x}{a} = \frac{y}{b}$$

and we can write $\frac{x}{a} = \frac{y}{b} = k$, where k is a constant,

$$\therefore\ x = ak \text{ and } y = bk$$

Example 3: If $3x = 5y$, find the value of the ratio $x : y$.

$$3x = 5y$$

Divide both sides by $3y$

$$\therefore \frac{x}{y} = \frac{5}{3}.$$

Example 4: If $4x = 3y$, find the value of $\frac{4x^2 - xy + y^2}{2x^2 + y^2}$.

Method I.

$$4x = 3y$$

$$\therefore \frac{x}{y} = \frac{3}{4}$$

$$\therefore \frac{4x^2 - xy + y^2}{2x^2 + y^2} = \frac{4\left(\frac{x^2}{y^2}\right) - \left(\frac{x}{y}\right) + 1}{2\left(\frac{x^2}{y^2}\right) + 1}$$ (Dividing numerator and denominator by y^2.)

$$= \frac{4(\frac{3}{4})^2 - (\frac{3}{4}) + 1}{2(\frac{3}{4})^2 + 1}$$

$$= \frac{\frac{9}{4} - \frac{3}{4} + 1}{\frac{9}{8} + 1}$$

$$= \frac{\frac{10}{4}}{\frac{17}{8}}$$

$$= \frac{10}{4} \times \frac{8}{17}$$

$$= \frac{20}{17}$$

Method II

$$4x = 3y$$

$$\therefore \frac{x}{3} = \frac{y}{4}$$

Let each ratio $= k$, a constant

$$\therefore \frac{x}{3} = \frac{y}{4} = k$$

$$\therefore x = 3k;\ y = 4k$$

$$\therefore \frac{4x^2 - xy + y^2}{2x^2 + y^2} = \frac{4(3k)^2 - (3k)(4k) + (4k)^2}{2(3k)^2 + (4k)^2}$$
$$= \frac{36k^2 - 12k^2 + 16k^2}{18k^2 + 16k^2}$$
$$= \frac{40k^2}{34k^2}$$
$$= \frac{20}{17}$$

Example 5: If $2(x^2 + y^2) = 5xy$, find the values of the ratio $\frac{x}{y}$.

$$2(x^2 + y^2) = 5xy$$
$$\therefore 2x^2 + 2y^2 = 5xy$$
$$\therefore 2x^2 - 5xy + 2y^2 = 0$$
$$\therefore (2x - y)(x - 2y) = 0$$
$$\therefore 2x = y \text{ or } x = 2y$$
$$\therefore \frac{x}{y} = \frac{1}{2} \text{ or } \frac{x}{y} = \frac{2}{1}$$

Example 6: Two numbers are in the ratio 4 : 7. If each number is increased by 4, the ratio becomes 2 : 3. Find the numbers.

[*Note:* If x and y are the numbers, then

$$\frac{x}{y} = \frac{4}{7}$$
$$\therefore \frac{x}{4} = \frac{y}{7}$$
$$\text{Let } \frac{x}{4} = \frac{y}{7} = k$$
$$\therefore x = 4k \text{ and } y = 7k.]$$

Let the numbers be $4k$ and $7k$.

$$\therefore \frac{4k + 4}{7k + 4} = \frac{2}{3}$$
$$\therefore 14k + 8 = 12k + 12$$
$$\therefore 2k = 4$$
$$\therefore k = 2$$

$\therefore$ The numbers are 8 and 14.

Exercises 77

Find the ratio of:

1. 13s. 4d. to 16s. 8d.
2. $15a$ shillings to £$3a$
3. 48 lb. to $\frac{1}{2}$ cwt.
4. $2n$ cwt. to $168n$ lb.
5. 96 sq. in. to $\frac{3}{4}$ sq. ft.
6. a^2 sq. in. to ab sq. ft.
7. 2s. 6d. per oz. to £1 10s. per lb.
8. 2s. 1d. per ft. to 7s. 6d. per yd.
9. £22 per mile to £1 10s. per 100 yd.
10. 60 m.p.h. to 2200 yd. per min.
11. 45 m.p.h. to 60 ft. per sec.
12. a m.p.h. to b ft. per sec.
13. If $2x = 7y$, find the ratio $x : y$.
14. If $\dfrac{2x - y}{x + 2y} = \frac{3}{4}$, find the ratio $x : y$.
15. If y is 50% greater than x, find the ratio $x : y$.
16. If $3x = 8y$, find the value of $\dfrac{2x + 3y}{x - y}$.
17. If $3x = 2y$, prove that $\dfrac{4x - y}{x + y} = \dfrac{x + 2y}{4x}$.
18. If $\dfrac{x}{y} = \frac{2}{3}$, find the values of the ratios:

(i) $\dfrac{2x - y}{x + y}$, (ii) $\dfrac{x^2 - y^2}{x^2 + y^2}$, (iii) $\dfrac{x^2}{xy + y^2}$

(iv) $\dfrac{ax}{by}$ (v) $\dfrac{ax + by}{bx + ay}$ (vi) $\left(\dfrac{x}{a}\right)^2 : \left(\dfrac{y}{b}\right)^2$

19. If $x^2 - 3xy + 2y^2 = 0$, find the values of x/y.
20. If $\dfrac{3x + y}{x + 2y} = \dfrac{3y}{4x}$, find the values of x/y.
21. The line AB is x in. long. The point P divides it in the ratio $2 : 3$. Find the lengths of AP and PB.
22. The line AB is x in. long. The point P divides it in the ratio $a : b$. Find the lengths of AP and PB.

23. The line AB is x in. long. The point P divides it externally in the ratio $a : b$. Find the lengths of AP and PB.

24. Two numbers are in the ratio 3 to 5. When 4 is added to each the ratio becomes 2 to 3. What are the numbers?

25. Two numbers are in the ratio 5 to 7. When each is diminished by 10 the ratio becomes 5 to 8. What are the numbers?

Proportion

Four numbers a, b, c, d are in proportion when

$$a : b = c : d \text{ or } \frac{a}{b} = \frac{c}{d}$$

The statement $\frac{a}{b} = \frac{c}{d}$ is called a Proportion.

The terms a and d are the *extremes* and the terms b and c are the *means* of the proportion

$$\text{Since } \frac{a}{b} = \frac{c}{d}$$

$$\therefore\ ad = bc$$

i.e., Product of extremes = Product of means

d is called the *Fourth Proportional* to a, b, c.

From the previous equation $d = \frac{bc}{a}$.

If $a : b = b : c$ or $\frac{a}{b} = \frac{b}{c}$, then b is called the *mean proportional* between a and c, and c is called the *third proportional* to a and b.

If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{e}{f}$ etc., then a, b, c, d, e, f, etc. are said to be in *continued proportion*.

When a, b, c, d are in proportion, certain relations can be readily deduced. To each relation a name has been given to enable reference to be made to them in the solutions of exercises.

$$\frac{a}{b} = \frac{c}{d} \qquad (1)$$

$$\therefore \frac{b}{a} = \frac{d}{c} \qquad \textit{Invertendo}$$

By cross-multiplication

$$ad = bc$$

Dividing throughout by cd,

$$\therefore \frac{a}{c} = \frac{b}{d} \qquad \textit{Alternando}$$

Adding 1 to both sides of equation (1),

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

$$\therefore \frac{a+b}{b} = \frac{c+d}{d} \quad . \quad . \quad . \quad \textit{Componendo}$$

Subtracting 1 from both sides of equation (1),

$$\frac{a}{b} - 1 = \frac{c}{d} - 1$$

$$\therefore \frac{a-b}{b} = \frac{c-d}{d} \quad . \quad . \quad . \quad \textit{Dividendo}$$

$$\therefore \frac{\frac{a+b}{b}}{\frac{a-b}{b}} = \frac{\frac{c+d}{d}}{\frac{c-d}{d}}$$

i.e. $$\frac{a+b}{a-b} = \frac{c+d}{c-d} \quad . \quad \textit{Componendo and dividendo}$$

The above relations can be proved by the following method, e.g. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Let $\frac{a}{b} = \frac{c}{d} = k$ (a constant).

$$\therefore\ a = bk;\ c = dk$$

$$\therefore\ \frac{a+b}{a-b} = \frac{bk+b}{bk-b} = \frac{b(k+1)}{b(k-1)} = \frac{k+1}{k-1}$$

$$\frac{c+d}{c-d} = \frac{dk+d}{dk-d} = \frac{d(k+1)}{d(k-1)} = \frac{k+1}{k-1}$$

$$\therefore\ \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

Example 1: Find the mean proportional between 18 and 8.
Let x be the mean proportional.

Then
$$\frac{18}{x} = \frac{x}{8}$$
$$\therefore\ x^2 = 144$$
$$\therefore\ x = \pm 12$$
Mean proportional is ± 12.

Example 2: Solve the equation $\dfrac{x^2+2x-3}{2x-3} = \dfrac{3x^2+2x-1}{2x-1}$

$$\frac{x^2+2x-3}{2x-3} = \frac{3x^2+2x-1}{2x-1}$$

By Dividendo,

$$\frac{x^2+2x-3-(2x-3)}{2x-3} = \frac{3x^2+2x-1-(2x-1)}{2x-1}$$

$$\therefore\ \frac{x^2}{2x-3} = \frac{3x^2}{2x-1}$$

$$\therefore\ 3x^2(2x-3) - x^2(2x-1) = 0$$

$$\therefore\ x^2(6x-9-2x+1) = 0$$

$$\therefore\ x^2(4x-8) = 0$$

$$\therefore\ x = 0,\ \text{(twice) or } x = 2$$

Example 3: If $(2a+c+4b+2d)(2a-c-4b+2d)$
$= (2a+c-4b-2d)(2a-c+4b-2d)$,
prove that a, b, c, d are in proportion.

We may write the given equation in the form

$$\frac{2a + c + 4b + 2d}{2a + c - 4b - 2d} = \frac{2a - c + 4b - 2d}{2a - c - 4b + 2d}$$

By Componendo and Dividendo,

$$\frac{4a + 2c}{8b + 4d} = \frac{4a - 2c}{8b - 4d}$$

$$\therefore \frac{2(2a + c)}{2(4b + 2d)} = \frac{2(2a - c)}{2(4b - 2d)}$$

By Alternando,

$$\frac{2a + c}{2a - c} = \frac{4b + 2d}{4b - 2d}$$

$$\frac{2a + c}{2a - c} = \frac{2b + d}{2b - d}$$

By Componendo and Dividendo,

$$\frac{4a}{2c} = \frac{4b}{2d}$$

$$\therefore \frac{a}{c} = \frac{b}{d}$$

By Alternando,

$$\frac{a}{b} = \frac{c}{d}$$

$\therefore$ a, b, c, d are in proportion

Exercises 78

1. Find x if the following sets of numbers are in proportion:

(a) $x, 3, 14, 21$ (c) $\frac{1}{3}, \frac{1}{4}, x, \frac{1}{6}$
(b) $7, x, 8, 2\frac{2}{3}$ (d) $x, 3, 8, 6x$

2. Find a fourth proportional to:

(a) 4, 6, 8 (b) $a, 2a, 3a$ (c) a, a^2, a^3

3. What number must be added to each of the numbers 1, 3, 5, 9 to give four numbers in proportion?

Solve the equations:

4. $\dfrac{x^2 + x + 1}{x + 1} = \dfrac{4x^2 + 3x + 5}{3x + 5}$

5. $\dfrac{2x^2 - x + 2}{x - 2} = \dfrac{3x^2 - 2x + 1}{2x - 1}$

6. $\dfrac{3x^2 - 2x + 3}{3x^2 + 2x - 3} = \dfrac{2x^2 - x + 4}{2x^2 + x - 4}$

7. If $(3a - b) : (3c - d) = (4a + b) : (4c + d)$, prove that a, b, c, d are in proportion.

8. If $(5a + b) : (5b + c) = (3a - 2b) : (3b - 2c)$, prove that $a : b = b : c$.

9. If $\dfrac{a + b + c + d}{a + b - c - d} = \dfrac{a - b + c - d}{a - b - c + d}$, prove that $\dfrac{a}{b} = \dfrac{c}{d}$.

10. If $\dfrac{a + b + c + d}{b + d} = \dfrac{c + d}{d}$, prove that $\dfrac{a}{b} = \dfrac{c}{d}$.

11. Find the third proportional to 6 and 18.

12. Find the mean proportional between 8 and 32.

13. Three numbers x, 24, 36 are in continued proportion. Find x.

14. Three positive numbers are in continued proportion. The second exceeds the first by 8, and the third is nine times the first. Find the numbers.

15. If a, b, c, d are in continued proportion, prove that $ac - bd = (b - c)(b + c)$.

Chapter 25

VARIATION I

Direct Variation

If one quantity y assumes the values $y_1, y_2, y_3 \ldots$ when another quantity, x, assumes the corresponding values $x_1, x_2, x_3 \ldots$, and if

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \ldots$$

it is said that y varies directly as x, or y varies as x, and we write

$y \propto x$, which is read "y varies as x"

The following table shows corresponding values of two quantities, s, a distance, and t, a time, for a body moving at a certain speed:

t	10	20	30	40	50
s	6	12	18	24	30

t is measured in minutes and s is measured in miles.

If we calculate in each case the ratio of the number of units in s to the number of units in t we find that in all cases

$$\frac{s}{t} = \frac{3}{5}$$

and therefore, we say $s \propto t$. The ratio $\frac{s}{t}$ is constant no matter what units we use.

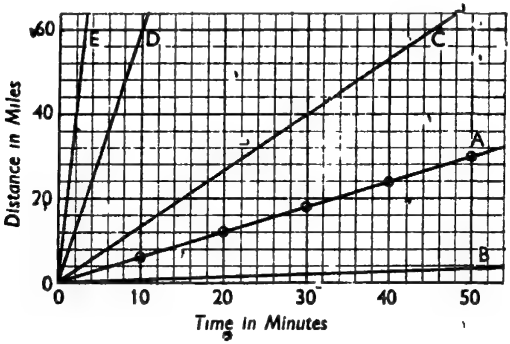
If s is measured in miles and t is measured in hours, we find that the ratio $\frac{s}{t}$ is constant but now $\frac{s}{t} = \frac{3}{5/60} = 36$.

The formula connecting s, the distance travelled, with t, the time and v, the speed, is

$$s = vt$$

$$\text{i.e. } \frac{s}{t} = v$$

In the above case $\frac{s}{t} = 36$, using the mile and the hour as units, i.e. the distances were travelled at a steady speed of 36 m.p.h.



When the above values of s and t are plotted on squared paper we get the straight-line graph OA, which passes through the origin.

The value of $\frac{s}{t}$ is the gradient of the graph, and as this ratio remains constant, the gradient is constant and the graph is a straight line.

Four other graphs are shown in the same diagram, each with a different value of the *Constant of Proportionality*.

The graph OB has a constant 4 and shows distances travelled in different times when the speed is 4 m.p.h.—a walking pace.

OC, where the constant is 80, might be the graph for a fast car, OD, where the constant is 360, might be the graph for an aeroplane and OE, for a very fast jet-plane.

In all cases $\frac{s}{t} = k$, a constant. Hence if $s \propto t$

$$\therefore \frac{s}{t} = k \text{ or } s = kt$$

When the value of t is doubled the value of s is doubled; if the value of s is reduced in the ratio 1 : 3 the value of t is reduced in the ratio 1 : 3. In fact, if the value of either s or t is increased or decreased in any ratio, the value of the other is increased or decreased in the same ratio.

The following table shows values of s and corresponding values of t for a train, as it gathers speed after starting from rest:

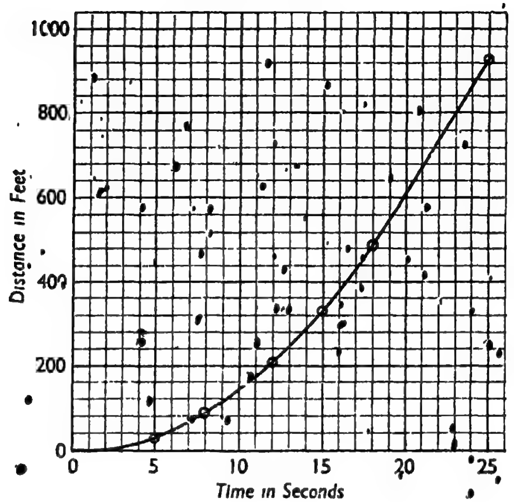
t	5	8	12	15	18	25
s	37 5	96	216	337 5	486	937 5

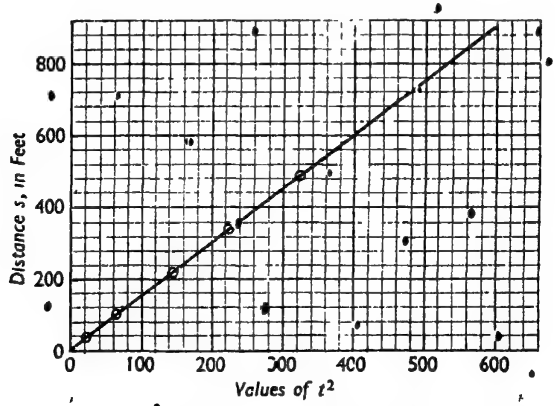
t is measured in seconds and s in feet.

When the values of s are plotted against the corresponding values of t, we get the curve shown on p. 257. Here, the ratio $\frac{s}{t}$ is not constant and does not vary as t.

If, however, values of s are plotted against values of t^2, we get the following table:

t^2	25	64	144	225	324	625
s	37 5	96	216	337 5	486	937 5





The graph is a straight line, indicating that s varies as t^2, i.e. $s \propto t^2$.

$$\therefore \frac{s}{t^2} = k \text{ or } s = kt^2$$

From the values given,

$$\frac{s}{t^2} = \frac{3}{2}$$

When $y \propto x^2$ and the value of x is multiplied by 2, 3, 4, etc., the corresponding value of y is found by multiplying by 4, 9, 16, etc.

e g. if x_1, y_1 and x_2, y_2 are corresponding values of x and y

$$y_1 = kx_1^2 \text{ (where } k \text{ is a constant)}$$
$$y_2 = kx_2^2$$

If also $x_2 = 3x_1$,

$$\therefore \text{ then } y_2 = k(3x_1^2)$$
$$= 9kx_1^2$$

i.e. when the value of x is multiplied by 3 the corresponding value of y is found by multiplying by 9.

The area of a circle varies as the square of its radius, i.e.

$$A \propto r^2$$
$$\therefore A = r^2$$

and, in fact, we know that $k = \pi$.

$$\therefore A = \pi r^2$$

When the radius is doubled, trebled, etc., the area is multiplied by 4, 9, etc.

Other forms of direct variation arise.

If y varies as the square root of x, then

$$y \propto \sqrt{x}$$
$$\therefore y = k\sqrt{x} \text{ (where } k \text{ is a constant)}$$

Hence, if the value of x is doubled, the value of y is multiplied by $\sqrt{2}$.

The volume of a sphere varies as the cube of the radius, i.e.

$$V \propto r^3$$

$$\therefore V = kr^3 = \tfrac{4}{3}\pi r^3$$

If the radii of two spheres are in the ratio 2 : 1, their volumes are in the ratio 8 : 1.

Inverse Variation

If x and y are connected by a relation which is such that when x is doubled y is halved; when y is trebled, x takes one-third of its previous value, and, in general, if x is increased in the ratio $\frac{p}{q}$, y is decreased in the ratio $\frac{q}{p}$, then y is said to vary inversely as x.

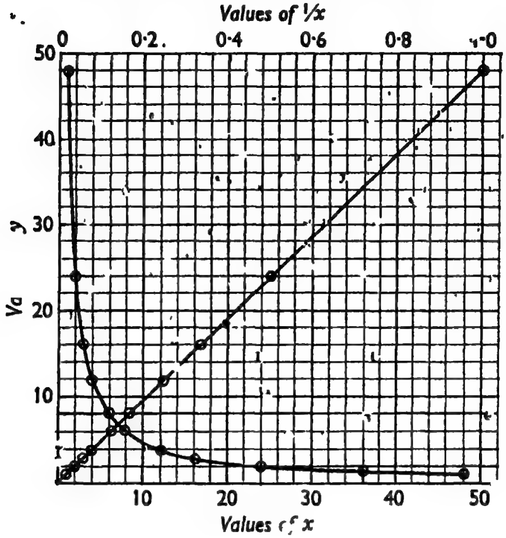
E.g. 48 articles at 1*d*. each may be bought for 4*s*., or 24 articles at 2*d*. each or 16 articles at 3*d*. each and so on. If x denotes the price in pence of each article and y denotes the number of such articles that can be bought for 4*s*. we get the following table:

x	48	24	16	12	8	6	4	3	2	1
y	1	2	3	4	6	8	12	16	24	48

When values of y are plotted against values of x we obtain the curve shown in the diagram on p. 260.

If, for each value of y, we calculate the corresponding value of $\frac{1}{x}$ we get the following table:

$\frac{1}{x}$	·021	·042	·063	·083	·125	·167	·250	·333	·5	1
y	1	2	3	4	6	8	12	16	24	48



When values of y are plotted against values of $\frac{1}{x}$ we get the straight-line graph, shown on the same diagram.

$$\therefore \quad \frac{y}{1/x} \doteq \text{constant}$$

$$\therefore \quad y = k \cdot \frac{1}{x}$$

$$= \frac{k}{x},$$

the value of k, the constant, being, in this case, 48.

Other forms of inverse variation may arise, e.g. the Inverse Square Law, which occurs in several branches of Physics.

The intensity of illumination (I) at a distance (d) from a

source of light varies inversely as the square of the distance (d), i.e.

$$I \propto \frac{1}{d^2}$$

$$\therefore\ I = \frac{k}{d^2}.$$

If I_1 and I_2 are the intensities at distances d_1 and d_2 from the source, then

$$I_1 = \frac{k}{d_1^2} \text{ and } I_2 = \frac{k}{d_2^2}$$

To compare the intensities

$$\frac{I_2}{I_1} = \frac{k/d_2^2}{k/d_1^2} = \left(\frac{d_1}{d_2}\right)^2$$

If $d_2 = 2d_1$, then

$$\frac{I_2}{I_1} = \left(\frac{d_1}{2d_1}\right)^2 = \frac{1}{4}$$

$$\therefore\ I_2 = \tfrac{1}{4}I_1$$

I.e. when the distance is doubled, the intensity is only $\frac{1}{4}$ of its previous value.

Similarly, when the distance is halved, the intensity is multiplied by 4.

Example 1: If y has the value 6 when x has the value 8, calculate the value of y when x has the value 48:

(i) when y varies as x;
(ii) when y varies as x^2;
(iii) when y varies inversely as x.

(i) $y \propto x$.
$\therefore\ y = kx$, where k is a constant, whose value is to be found.
When $x = 8$, $y = 6$

$$\therefore\ 6 = 8k$$

$$\therefore\ k = \tfrac{3}{4}$$

Substituting this value of k in the equation $y = kx$.

$$\therefore y = \frac{3x}{4}$$

When $x = 48$, $y = \dfrac{3 \times 48}{4}$

$$= 36$$

(ii) $y \propto x^2$.

$$\therefore y = kx^2$$

When $x = 8$, $y = 6$.

$$\therefore 6 = 64k$$

$$\therefore k = \frac{3}{32}$$

$$\therefore y = \frac{3x^2}{32}$$

When $x = 48$, $y = \dfrac{3 \times 48 \times 48}{32}$

$$= 216$$

(iii) $y \propto \dfrac{1}{x}$.

$$\therefore y = \frac{k}{x}$$

When $x = 8$, $y = 6$

$$\therefore 6 = \frac{k}{8}$$

$$\therefore k = 48$$

$$\therefore y = \frac{48}{x}$$

When $x = 48$.

$$\therefore y = \frac{48}{48}$$

$$= 1$$

Example 2: The time of swing of a pendulum varies as the square root of its length.

For a pendulum $2\frac{1}{4}$ ft. long the time of swing is $1\frac{2}{3}$ sec. Calculate the time of swing for a pendulum $6\frac{1}{4}$ ft. long.

Let T sec. be the time of swing, and l ft. be the length of the pendulum.

$$\text{Then } T \propto \sqrt{l}$$
$$\therefore T = k\sqrt{l}$$

When $l = 2\frac{1}{4}$, $T = 1\frac{2}{3}$,

$$\therefore 1\frac{2}{3} = k\sqrt{2\frac{1}{4}}$$
$$\therefore \frac{5}{3} = k \cdot \frac{3}{2}$$
$$\therefore k = \frac{10}{9}$$
$$\therefore T = \frac{10}{9}\sqrt{l}$$

When $l = 6\frac{1}{4}$,

$$T = \frac{10}{9}\sqrt{6\frac{1}{4}} = \frac{10}{9} \cdot 2\frac{1}{2}$$
$$= \frac{25}{9}$$
$$= 2\frac{7}{9}$$
$$\therefore \text{Time of swing} = 2\frac{7}{9} \text{ sec.}$$

Exercises 79

1.

x	2	5	8	20	26
y	0·4	1	1·6	4	5·2

Examine the values given for x and y and write down:

(1) value of y when x is 2·5;
(2) value of x when y is 2·5.

What kind of graph is obtained by plotting values of y against those of x?

Find the constant of proportionality and write down the equation connecting y and x.

2.

x	2	3	5	6	8	11
y	1	$2\frac{1}{4}$	$6\frac{1}{4}$	9	16	$30\frac{1}{4}$

Graphically, or otherwise, show that the above values of x and y satisfy the relation $y \propto x^2$.

Find the constant of proportionality and write down:

(i) value of y when x is $\frac{1}{2}$;
(ii) the positive value of x when $y = \frac{1}{4}$.

3.

x	3	5	8	12	15	20
y	40	24	15	10	8	6

Is the variation direct or inverse?

Deduce from the above values the equation connecting x and y.

4. The volume (v) of a mass of gas was measured at different pressures (p), and the following results were obtained. p is measured in cm. of mercury and v is measured in c.c.

p	100	85	76	64	50	36
v	24	28·2	31·6	37·5	48	66·7

Calculate corresponding values of $\frac{1}{v}$, and, allowing for experimental errors, show that p and v are connected by the relation $p = k \cdot \frac{1}{v}$. Find the value of k.

In Exercises 5–8 find the law of variation in each case and express y in terms of x.

5.

x	1	2	4	6	10
y	$\frac{1}{4}$	1	4	9	25

6.

x	1	1·44	2·25	4	6·25
y	12	14·4	18	24	30

7.

x	$\frac{1}{4}$	1	9	16	100
y	36	18	6	$4\frac{1}{2}$	$1\frac{4}{5}$

8.

x	1	2	5	8	10
y	120	30	4·8	1·875	1·2

9. y varies inversely as x. If $y = 6$, when $x = 4$, calculate the value of:

(i) y, when $x = 12$;
(ii) x, when $y = 48$.

10. y varies as x^2. If, when $x = 10$, $y = 10$, calculate the value of:

(i) y, when $x = 5$;
(ii) x, when $y = 160$.

11. y varies inversely as the square root of x.
If $y = 4$, when $x = 9$, calculate the value of:

(i) y, when $x = 1{\cdot}44$;
(ii) x, when $y = 15$.

12. y varies as the square root of x^3. If $y = 4$, when $x = 4$, calculate the value of y when $x = 16$, and the value of x when $y = 13\frac{1}{2}$.

13. The pressure of the water at any point below the surface of the sea varies directly as the depth of the point below the surface.

If the pressure at a depth of 33 ft. is 15 lb. per sq. in., calculate the pressure at a depth of 49·5 ft.

14. The number of articles that can be bought for a given sum of money varies inversely as the price of each article. If 72 articles can be bought when each article costs £3 5*s.*, how many more can be bought for the same sum when the price drops to £3 each?

15. The pressure of a given mass of gas varies inversely as the volume so long as the temperature remains constant.

If a mass of gas has a volume of 450 c.c. when the pressure is 76 cm. of mercury, what increase of pressure (in cm. of mercury) will be necessary to compress the gas by 70 c.c.?

16. The force of attraction, F, between two unlike magnetic poles varies inversely as the square of the distance, d, between them. When the poles are 10 cm. apart the force is 8·4 gm. weight. What will the force be if the poles are 7 cm. apart?

If the distance is increased in the ratio 3 : 2, in what ratio is the force decreased?

17. When a body falls vertically from rest the distance, d, through which it falls varies as the square of the time, t, of falling.

If, in the first 2 sec. of its motion the body falls 64 ft., through what distance will it fall in the next 2 sec.?

18. The volumes of pyramids of equal height vary as the areas of the bases on which they stand.

If the volume of a pyramid on a square base of side 6 ft. is 108 cu. ft., calculate the volume of a pyramid of the same height whose base is an equilateral triangle of side 6 ft.

19. The frequency, N, of tuning forks with prongs of similar cross-section varies inversely as the square of the length, l, of the prongs.

If the frequency is 256 when the prongs are 3·6 in. long, calculate the length of the prongs of a tuning fork whose frequency is 324.

20. The value of diamonds of the same quality varies as the square of their weight. The largest diamond ever found, the Cullinan Diamond, weighed 3000 carats and was valued at £15,000,000. What would be the value of a similar diamond weighing 150 carats?

If this diamond were cut into two parts whose weights were in the ratio 3 : 2, what would be the total loss in value?

21. Water flows at a steady rate into a tank so shaped that the depth d ft varies as the square root of the time, t min., during which the water has been flowing.

If, after 4 min., the depth is 8 in., how much longer must the water flow to double the depth? By how much is the depth increased during the second 4 min.?

22. The receipts from a certain air journey vary as the excess of the average speed over 240 m p.h., and the expense of flying the aircraft varies as the square of this excess. The receipts and expense are equal when the aircraft flies at an average speed of 300 m.p.h. Find the average speed when the profits are one-third of the receipts.

23. The square of the time T of a planet's revolution round the sun varies as the cube of its distance D millions of miles from the sun. The earth, which is $91\frac{1}{4}$ millions of miles from the sun, completes one revolution round the sun in 365 days. Find the time of Venus's revolution, given that Venus is 66 millions of miles from the sun.

Joint Variation

If z varies as the product of x and y, then z is said to vary jointly as x and y.

$$z \propto xy$$
$$\therefore\ z = kxy \text{ (where } k \text{ is a constant)}$$

Example 1: If A varies jointly as b and h, and $A = 24$ when $b = 6$ and $h = 8$, find A when $b = 10$ and $h = 7$.

$$\text{Since } A \propto bh$$
$$\therefore\ A = kbh \text{ (where } k \text{ is a constant)}$$
$$\therefore\ 24 = k \,.\, 6 \,.\, 8$$
$$\therefore\ k = \frac{24}{6 \,.\, 8} = \tfrac{1}{2}$$
$$\therefore\ A = \tfrac{1}{2}bh$$

When $b = 10$, $h = 7$,

$$A = \tfrac{1}{2} \,.\, 10 \,.\, 7 = 35$$

Example 2: The volume of a cylinder varies jointly as the square of the radius and the height. If the volume is 308 cu. in. when the radius is 7 in. and the height is 2 in., find the volume when the radius is $3\frac{1}{2}$ in. and the height is 10 in.

Let V cu. in., r in. and h in. represent the volume, the radius and the height respectively.

$$\text{Since } V \propto r^2h$$
$$\therefore\ V = kr^2h \text{ (where } k \text{ is a constant)}$$
$$\therefore\ 308 = k \,.\, 7^2 \,.\, 2$$
$$\therefore\ k = \frac{308}{7^2 \,.\, 2} = \frac{22}{7}$$
$$\therefore\ V = \tfrac{22}{7} \,.\, r^2h$$

When $r = 3\frac{1}{2}$, $h = 10$,

$$\therefore\ V = \frac{22}{7} \,.\, \left(\frac{7}{2}\right)^2 \,.\, 10 = 385$$
$$\therefore\ \text{Volume} = 385 \text{ cu. in.}$$

If z varies directly as x and inversely as y, i.e. $z \propto \frac{x}{y}$, then $z = k \cdot \frac{x}{y}$ (when k is a constant).

Example 3: If W varies directly as b and inversely as l, and $W = 12$ when $b = 2$, $l = 10$, find W when $b = 1\frac{1}{2}$; $l = 15$.

$$W \propto \frac{b}{l}$$

$$\therefore \quad W = k \cdot \frac{b}{l} \text{ (where } k \text{ is a constant)}$$

$$\therefore \quad 12 = k \cdot \frac{2}{10}$$

$$\therefore \quad k = 60$$

$$\therefore \quad W = 60 \cdot \frac{b}{l}$$

When $b = 1\frac{1}{2}$, $l = 15$, $W = 60 \times \frac{1\frac{1}{2}}{15} = 6$.

Example 4: The resistance of a cylindrical wire varies directly as its length and inversely as the square of its radius. If the resistance is 10 units when the length is 100 yd. and the radius is $\frac{1}{20}$ in., find the length of wire required of radius $\frac{1}{10}$ in. to give a resistance of 22 units.

Let R units be the resistance, l yd. = the length, r in. = radius.

$$R \propto \frac{l}{r^2}$$

$$\therefore \quad R = k \cdot \frac{l}{r^2} \text{ (where } k \text{ is a constant)}$$

$$\therefore \quad 10 = k \cdot \frac{100}{(\frac{1}{20})^2}$$

$$\therefore \quad k = \frac{10 \cdot (\frac{1}{20})^2}{100} = \frac{1}{4000}$$

$$\therefore \quad R = \frac{1}{4000} \cdot \frac{l}{r^2}$$

When $R = 22$, $r = \frac{1}{10}$,

$$22 = \frac{1}{4000} \cdot \frac{l}{(\frac{1}{10})^2}$$

$$\therefore\ l = 4000 \times 22 \times (\tfrac{1}{10})^2$$

$$= 880$$

$$\therefore \text{Length} = 880 \text{ yd.} = \tfrac{1}{2} \text{ mile}$$

Exercises 80

1. If x varies jointly as y and z, and $x = 80$ when $y = 8$, $z = 5$, find x when $y = 12$, $z = 2$.

2. If x varies jointly as y and z, and $x = 96$ when $y = 6$, $z = 4$, find y when $x = 60$, $z = 5$.

3. If E varies directly as m and inversely as l, and $E = 12$ when $m = 8$, $l = 6$, find E when $m = 10$, $l = 9$.

4. If z varies jointly as y and the square of x, and $z = 12$ when $y = 4$, $x = 3$, find z when $y = 6$, $x = 5$.

5. If l varies directly as x and inversely as the cube of y, and $l = 1\frac{1}{2}$ when $x = 18$, $y = 2$, find l when $x = 81$, $y = 3$.

6. The area of the curved surface of a cylinder varies jointly as the radius of the base and the height, and the area is 314 sq. in. when the radius is 5 in. and the height 10 in. Find the area of the curved surface if the radius is 3 in. and the height 5 in.

7. The area of an ellipse varies jointly as the lengths of its axes. If the area is 155 sq. cm. when the axes measure 10 cm. and 20 cm. respectively, find the area when the lengths of the axes are 8 cm. and 6 cm. respectively.

8. The pressure at any point in a liquid varies jointly as the depth of the point below the surface of the liquid and the density of the liquid. If the pressure is 24 units at a point 2 ft. below the surface of a liquid of density 50 lb. per cu. ft., what would the pressure be at a depth of 3 ft. in a liquid of density 75 lb. per cu. ft.?

9. For a given mass of gas the volume varies directly as the absolute temperature and inversely as the pressure. The volume of a certain mass of gas is 40 c.c. when the absolute temperature is 300° and the pressure 15 units. What would the volume become if the absolute temperature rose to 350° and the pressure decreased to 10 units?

10. The distance a body travels in a certain time from rest when uniformly accelerated varies jointly as the acceleration and the square of the time. If the distance travelled in 6 sec. is 180 ft. when the acceleration is 10 ft. per sec. per sec., find the distance travelled in 8 sec. when the acceleration is 4 ft. per sec. per sec.

CHAPTER 26

INDICES I

IN Chapter 1, a^4 was defined as the product of four factors, each of which was a.

i.e. $$a^4 = a \times a \times a \times a$$

In a later chapter it was shown that:

(1) $a^4 \times a^5 = a^{4+5} = a^9$,

i.e. that **to multiply powers of the same number, the indices must be added. This is the First Law of Indices.** It was also shown that:

(2) $\dfrac{a^6}{a^2} = a^{6-2} = a^4$

(3) $(a^2)^3 = a^{2 \times 3} = a^6$

(4) $(ab^2)^2 = (a)^2(b^2)^2 = a^2b^4$

Examples (2), (3) and (4) illustrate the Second, Third and Fourth Laws of Indices.

So far all indices have been positive whole numbers, and, since a power is defined as the product of a certain number of equal factors, it would appear that an index must be a positive whole number. It has, however, been found convenient to use as indices not only positive whole numbers, but also zero, i.e. 0, negative numbers, e.g. -2, and fractions, e.g. $\frac{1}{2}$, $-\frac{2}{3}$. We must therefore have meanings for expressions such as a^0, 3^{-2}, $5^{\frac{1}{2}}$, $8^{-\frac{2}{3}}$. These meanings cannot come from the definition of a power, for we cannot think of a number of equal factors being 0, -2, $\frac{1}{2}$ or $-\frac{2}{3}$. Intelligible meanings, however, can be found if we assume first of all that the First Law of Indices holds good for all indices, whether they are

positive or negative, whole numbers or fractions. **We assume that to multiply powers of the same number the indices are added, no matter what the indices are.**

We can now proceed as follows:

1. *To Find a Meaning for* a^0

$$a^0 \times a^2 = a^{0+2} \text{ (applying the First Law of Indices)}$$
$$= a^2$$

Divide both sides by a^2.

$$\therefore a^0 = 1$$

Hence $5^0 = 1$, $(-3)^0 = 1$, $(\frac{2}{5})^0 = 1$.

2. *To Find a Meaning for* 3^{-2}

$$3^{-2} \times 3^2 = 3^{-2+2} \text{ (applying the First Law of Indices)}$$
$$= 3^0$$
$$= 1$$

Divide both sides by 3^2.

$$\therefore 3^{-2} = \frac{1}{3^2}$$

Hence $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.

$$(\tfrac{1}{3})^{-2} = \frac{1}{(\frac{1}{3})^2} = \frac{1}{\frac{1}{9}} = 9$$

3. *To Find Meaning for* $5^{\frac{1}{2}}$

$$5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} \text{ (applying the First Law of Indices)}$$
$$= 5^1$$
$$= 5$$

Take the square root of each side.

$$\therefore 5^{\frac{1}{2}} = \sqrt{5}$$

Similarly, to find a meaning for $7^{\frac{2}{3}}$,

$$7^{\frac{2}{3}} \times 7^{\frac{2}{3}} \times 7^{\frac{2}{3}} = 7^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}}$$
$$= 7^2$$

Take the cube root of each side

$$7^{\frac{2}{3}} = \sqrt[3]{7^2}$$

Hence $16^{\frac{1}{2}} = \sqrt{16} = 4$ (the square root of 16 is 4 or -4, but throughout this chapter only the positive value of the square root is used).

$$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

It is important to remember that, once the First Law of Indices has been assumed to be true for all indices, and we have found the above meanings for zero index, and for negative and fractional indices, all the laws of indices can now be used for all indices, positive or negative, whole numbers or fractions.

Example 1: Simplify $a^5 \times a^{-3} \times a^{-2}$.

$$\begin{aligned} a^5 \times a^{-3} \times a^{-2} &= a^{5-3-2} \\ &= a^0 \end{aligned}$$

Example 2: If $a = 4$, find the value of $a^{\frac{3}{2}}$.

$$\begin{aligned} a^{\frac{3}{2}} &= 4^{\frac{3}{2}} \\ &= (2^2)^{\frac{3}{2}} \\ &= 2^3 \\ &= 8 \end{aligned}$$

or $a^{\frac{3}{2}} = 4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$

Example 3: Express $8^{-\frac{2}{3}} \times 4^{\frac{1}{2}} \times 16$ as a power of 2.

$$\begin{aligned} 8^{-\frac{2}{3}} \times 4^{\frac{1}{2}} \times 16 &= (2^3)^{-\frac{2}{3}} \times (2^2)^{\frac{1}{2}} \times 2^4 \\ &= 2^{-2} \times 2^1 \times 2^4 \\ &= 2^{-2+1+4} \\ &= 2^3 \end{aligned}$$

Example 4: Simplify $\dfrac{2^{\frac{1}{3}} \div 6^{-\frac{2}{3}}}{\sqrt[3]{9}}$.

$$\begin{aligned}
\frac{2^{\frac{1}{3}} \div 6^{-\frac{2}{3}}}{\sqrt[3]{9}} &= \frac{2^{\frac{1}{3}} \div (2 \times 3)^{-\frac{2}{3}}}{9^{\frac{1}{3}}} \\
&= \frac{2^{\frac{1}{3}} \div 2^{-\frac{2}{3}}\, 3^{-\frac{2}{3}}}{(3^2)^{\frac{1}{3}}} \\
&= \frac{2^{\frac{1}{3}}}{2^{-\frac{2}{3}} . 3^{-\frac{2}{3}}} \times \frac{1}{3^{\frac{2}{3}}} \\
&= \frac{2^{\frac{1}{3}+\frac{2}{3}}}{3^0} \\
&= \frac{2}{1} \\
&= 2
\end{aligned}$$

Example 5: Simplify $\dfrac{(4a^2)^{\frac{3}{2}} \times a^{-2}}{2a^{\frac{1}{4}}}$ and find the value of the expression when $a = 16$.

$$\begin{aligned}
\frac{(4a^2)^{\frac{3}{2}} \times a^{-2}}{2a^{\frac{1}{4}}} &= \frac{4^{\frac{3}{2}} . (a^2)^{\frac{3}{2}} \times a^{-2}}{2 . a^{\frac{1}{4}}} \\
&= \frac{(2^2)^{\frac{3}{2}}(a^3) \times a^{-2}}{2 . a^{\frac{1}{4}}} \\
&= \frac{2^3 . a^3 . a^{-2}}{2a^{\frac{1}{4}}} \\
&= 2^{3-1} . a^{3-2-\frac{1}{4}} \\
&= 4a^{\frac{3}{4}}
\end{aligned}$$

When $a = 16$,

$$\begin{aligned}
4a^{\frac{3}{4}} &= 4 . 16^{\frac{3}{4}} \\
&= 4 . (2^4)^{\frac{3}{4}} \\
&= 4 . 2^3 \\
&= 4 \times 8 \\
&= 32
\end{aligned}$$

Exercises 81

Simplify:

1. $a^2 \times a^4 \times a$
2. $2a^2 \times a^3$
3. $6x^5 \div 3x^2$
4. $(y^2)^3$
5. $(2a^3b)^2$
6. $a^3 \times a^{-2} \times a$
7. $2a^4 \times 3a^{-1}$
8. $a^2 \div a^{-2}$
9. $\dfrac{6a^3}{3a^{-1}}$
10. $\dfrac{a^2 \times a^{-3}}{a^{-4}}$
11. $(a^{-1})^{-2}$
12. $(a^{-1}b^{-2})^{-1}$

Evaluate:

13. 10^0
14. 2^{-1}
15. $\dfrac{1}{3^{-2}}$
16. $\dfrac{2^{-2}}{3^{-[illegible]}}$
17. $(\frac{1}{2})^{-1}$
18. $(\frac{2}{3})^{-2}$
19. 10^{-3}
20. $\dfrac{2^{-1}}{2^{-2}}$
21. $2^0 \times 3^{-2}$
22. $(3^2)^0 \times 2^{-3}$
23. $\dfrac{2^3}{4^{-1}}$
24. $1{\cdot}8 \times 10^{-2}$
25. $\dfrac{2^{-3}}{12^{-1}}$
26. $(\frac{2}{3})^{-1} \times (\frac{1}{3})^0 \times (\frac{8}{27})^{\frac{1}{3}}$
27. $(\frac{1}{3})^{-1} \times (3^{-1})^{-2} \times (3^{-1})^3$

Simplify, expressing the answers with positive indices:

28. a^{-3}
29. $(a^{-1})^2$
30. $(a^{\frac{1}{2}}b^{-\frac{1}{2}})^2$
31. $(x^{-1})^{-1}$
32. $(2a)^{-2}$
33. $2a^{-2}$
34. $\left(\dfrac{1}{a}\right)^{-1}$
35. $(27y^{-6})^{\frac{1}{3}}$
36. $(16a^4b^{-2})^{-\frac{1}{2}}$
37. $3x^0$
38. $(3x)^0$
39. $\dfrac{a}{b^{-1}}$
40. $\dfrac{ab^{-1}}{a^{-1}b}$
41. $\left(\dfrac{a}{b}\right)^{-2}$
42. $(x^2)^{-3}$
43. $\left(\dfrac{1}{ab}\right)^{-2}$
44. $a^{-2}b^2$
45. $a^2b^{-1} \times a^{-2}b$
46. $(x^4)^{-\frac{1}{2}}$
47. $(x^{-\frac{1}{3}})^3$
48. $\dfrac{x^{-3}}{(2x)^2}$
49. $\dfrac{(3x^{-1})^2}{(x^2)^{-1}}$
50. $4a^{\frac{3}{2}} \times \sqrt{a} \div 2a^{-1}$
51. $(a^{-1})^{-1} \div (a^2)^{\frac{1}{2}}$

Evaluate:

52. $9^{\frac{1}{2}}$
53. $8^{\frac{1}{3}}$
54. $27^{\frac{2}{3}}$
55. $81^{\frac{3}{4}}$
56. $(\frac{1}{25})^{\frac{1}{2}}$
57. $4^{-\frac{1}{2}}$
58. $27^{-\frac{2}{3}}$
59. $9^{-\frac{1}{2}}$
60. $100^{-\frac{1}{2}}$
61. $(\frac{1}{9})^{-\frac{1}{2}}$
62. $(-8)^{-\frac{1}{3}}$
63. $(\frac{1}{16})^{-\frac{1}{4}}$
64. $(\frac{2}{3})^{-2}$
65. $(\frac{1}{8})^{-\frac{2}{3}}$
66. $64^{\frac{2}{3}} \times 16^{-\frac{1}{2}}$
67. $(\frac{4}{9})^{\frac{1}{2}} \times (\frac{1}{27})^{-\frac{1}{3}}$
68. $(\frac{1}{10})^{-1} \times (\frac{4}{5})^{\frac{1}{2}} \times 5^{\frac{1}{2}}$
69. $12^{\frac{2}{3}} \times (\frac{1}{9})^{-\frac{1}{2}} \times \sqrt[3]{\frac{3}{2}}$

70. Express with indices: $\sqrt{x}$, $\sqrt[3]{a}$, $\sqrt{a^3}$, $\sqrt[3]{b^2}$.

71. Express with root signs: $a^{\frac{1}{4}}$, $y^{\frac{1}{2}}$, $b^{\frac{2}{3}}$, $c^{\frac{1}{2}}$.

72. Express $18^{-\frac{1}{2}} \times 4^{\frac{1}{2}}$ as a vulgar fraction.

73. Express each of the following in the form $a \times b$ when a is a number between 1 and 10, and b is a power of 10:

(i) 3456
(ii) 0·286
(iii) 0·000043

74. If $y = 9^x$: (i) find y when $x = -2, \frac{1}{2}, -\frac{3}{2}$, and (ii) find x when $y = 9, \frac{1}{3}, \frac{1}{27}$.

75. Simplify $\dfrac{8^{-\frac{1}{3}} \times 18^{-\frac{1}{2}}}{3.2^{\frac{1}{2}}}$.

76. Evaluate $64^{\frac{1}{3}} + (\frac{1}{3})^{-2} + 16^{\frac{1}{2}}$.

77. Express $27^{\frac{2}{3}} \times 81^{-\frac{1}{2}} \div 9^{-1}$ as a power of 3.

78. Simplify $\dfrac{(3^4)^{-\frac{1}{2}} \div 18^{-1}}{2.12^{-1}}$.

79. Simplify:

(i) $0{\cdot}273 \times 10^2$
(ii) $3{\cdot}871 \times 10^{-3}$
(iii) 27×10^{-6}

80. Find the value of n in each case:

(i) $2^n = 32$ (ii) $3^n = \frac{1}{3}$ (iii) $2^n = 1$
(iv) $4^n = 8$ (v) $9^n = 243$

81. Simplify $\dfrac{a^{\frac{1}{2}} \times a^{\frac{1}{4}}}{(a^{\frac{1}{2}})^{\frac{1}{3}}}$ and evaluate the result when $a = 8$.

82. Simplify $\dfrac{(2a)^{-\frac{1}{2}} \times 2a^{\frac{1}{3}}}{\sqrt[3]{a}}$ and evaluate the result when $a = \frac{1}{4}$.

83. Simplify $(16a^2)^{-\frac{1}{4}} \times (27a^3)^{\frac{1}{3}}$ and evaluate the result when $a = \frac{4}{9}$.

Chapter 27

GRAPHS III—QUADRATIC AND CUBIC FUNCTIONS—THE FUNCTION m/x

Curve of Squares

$$y = x^2$$

Since x^2 is positive for all real values of x, it follows that y cannot take negative values and that no part of the graph can lie below the x-axis.

The following table gives corresponding values of x and y from $x = -5$ to $x = +5$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

The graph is shown in the diagram on p. 280, drawn to a scale of 1 in. to two units on the x-axis, and 1 in. to 4 units on the y-axis.

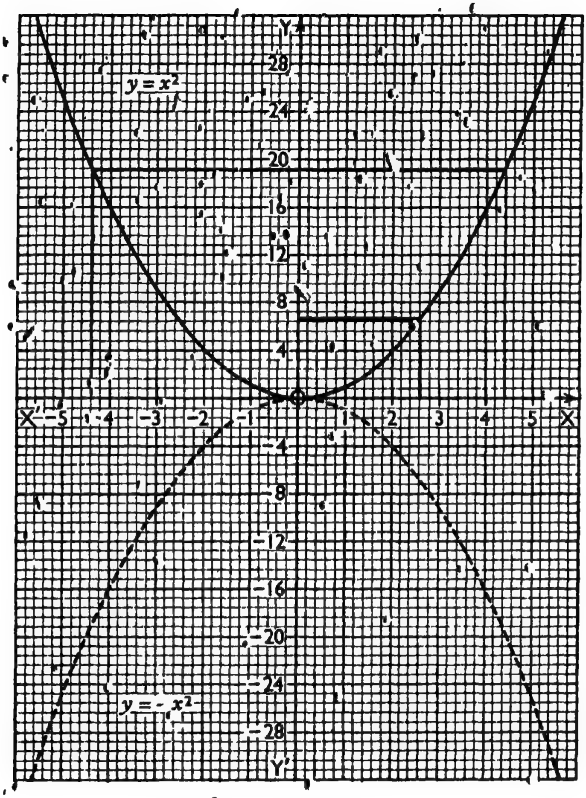
The graph is called a *Parabola*.

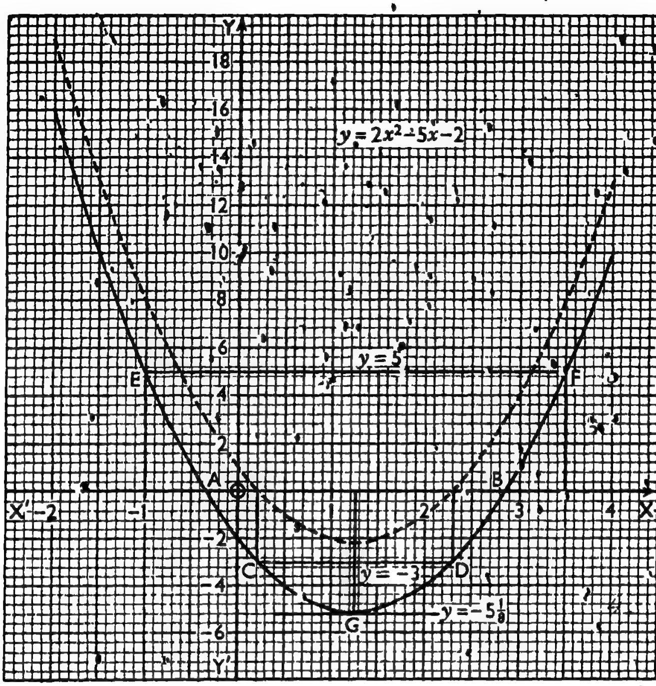
The graph can be used to determine the approximate square or square root of any number.

e.g. $(2{\cdot}6)^2 = 6{\cdot}7$ approximately
and $\sqrt{19} = \pm 4{\cdot}4$ approximately

Since x^2 is always positive, $-x^2$ is always negative. Thus the graph of $y = -x^2$ lies entirely below the x-axis.

The graph of $y = -x^2$ is shown as a dotted curve in the diagram.


Y
28
24
20
16
12
8
4
$y = x^2$
X′
−5
−4
−3
−2
−1
O
1
2
3
4
5
X
−4
−8
−12
−16
−20
−24
−28
Y′
$y = -x^2$



The Graph of the Quadratic Function $ax^2 + bx + c$ where a, b, c are constants.

Any algebraic expression whose value depends on the value of x is said to be a *function* of x.

Case I. Let the value of the function $2x^2 - 5x - 2$ be denoted by y i.e. $y = 2x^2 - 5x - 2$.

The following table shows corresponding values of x and y:

x	-2	-1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4
y	16	5	-2	-4	-5	-5	-4	-2	1	10

and the graph on p. 281 is drawn to a scale of 1 in to 1 unit on the x-axis and 1 in. to 4 units on the y-axis.

From the graph it is clear that the lowest point on the graph occurs at $x = 1\frac{1}{4}$.

When $x = 1\frac{1}{4}$, $y = -5\frac{1}{8}$, i.e. the minimum value of the function is $-5\frac{1}{8}$.

From the above graph, the graph of any function of the form $2x^2 - 5x + k$, where k is a constant, can be deduced.

e.g. Deduce the graph of $y = 2x^2 - 5x + 1$.

When $y = 2x^2 - 5x + 1$

we write $y = (2x^2 - 5x - 2) + 3$

Thus, for any given value of x, the value of y is greater by 3 than the corresponding value of y in the graph already drawn.

The required graph is therefore obtained by moving the original graph parallel to the y-axis through a distance equal to 3 units.

The dotted line in the figure is the graph of

$$y = 2x^2 - 5x + 1$$

The graph $y = 2x^2 - 5x - 2$ crosses the x-axis at two points A and B.

At all points on the x-axis the value of y is zero. Hence at A and B,

$$2x^2 - 5x - 2 = 0$$

and the values of x at A and B are the values of x which make $2x^2 - 5x - 2 = 0$.

At A, $x = -0{\cdot}35$ and at B, $x = 2{\cdot}85$.

The roots of the quadratic equation $2x^2 - 5x - 2 = 0$ are therefore $-0{\cdot}35$ and $2{\cdot}85$.

The graph $y = 2x^2 - 5x - 2$ may be used to find the roots of any quadratic equation of the form $2x^2 - 5x + k = 0$, where k is a constant.

e.g. Solve the equation $2x^2 - 5x + 1 = 0$.

When $2x^2 - 5x + 1 = 0$,

we write $(2x^2 - 5x - 2) + 3 = 0$,

i.e. $2x^2 - 5x - 2 = -3$

But $y = 2x^2 - 5x - 2$

$\therefore y = -3$

Thus, when $y = -3$, $2x^2 - 5x + 1 = 0$.

We therefore have to find the points on the graph at which y has the value -3 and read off the corresponding values of x.

$y = -3$ at the points C and D

The corresponding values of x are 0·2 and 2·3.

The roots of the equation $2x^2 - 5x + 1 = 0$ are therefore 0·2 and 2·3, a result which could also have been obtained from the dotted graph, using the points at which this graph cuts the x-axis.

Consider the equation $2x^2 - 5x - 2 = k$, where k is a constant.

To solve the equation $2x^2 - 5x - 2 = k$, since the graph is the graph of $y = 2x^2 - 5x - 2$, we put $y = k$, find the points on the graph at which $y = k$ and read off the corresponding values of x.

For all values of k, $y = k$ is a straight line parallel to the x-axis.

We have already shown that the minimum value of y is $-5\frac{1}{8}$. Hence for all values of k greater than $-5\frac{1}{8}$ the line $y = k$ will cut the graph in two points.

Thus, the line $y = 5$ cuts the graph in two points E and F and the values of x at these points are the roots of the equation

$$2x^2 - 5x - 2 = 5$$
$$\text{or } 2x^2 - 5x - 7 = 0$$

From the graph, the roots of this equation are -1 and $3\frac{1}{2}$, a result which may be checked algebraically, since, when

$$2x^2 - 5x - 7 = 0$$
$$\therefore (2x - 7)(x + 1) = 0$$
$$\therefore x = 3\tfrac{1}{2} \text{ or } x = -1$$

When $k = -5\frac{1}{8}$ the line $y = k$ touches the graph at its lowest point G, or cuts the graph in two coincident points. There is therefore only one value of x, $x = 1\frac{1}{4}$, which satisfies the equation

$$2x^2 - 5x - 2 = -5\tfrac{1}{8}$$

In this case we say that the two roots are equal.

When $2x^2 - 5x - 2 = -5\frac{1}{8}$.

$$\therefore 2x^2 - 5x + 3\tfrac{1}{8} = 0$$
$$\therefore 16x^2 - 40x + 25 = 0$$
$$\therefore (4x - 5)^2 = 0, \text{ which gives } x = 1\tfrac{1}{4} \text{ twice.}$$

When k is less than $-5\frac{1}{8}$, the line $y = k$ does not cut the graph.

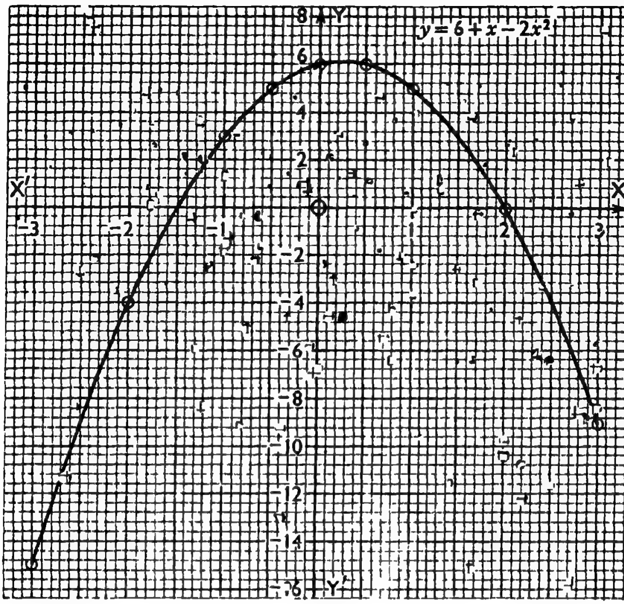
There are therefore no real values of x for which the function $2x^2 - 5x - 2$ can have a value less than $-5\frac{1}{8}$.

We say that the roots of the equation

$$2x^2 - 5x - 2 = k$$

where k is less than $-5\frac{1}{8}$, are imaginary.

Note carefully the difference between a quadratic *function* and a quadratic *equation*.



Case 2. Let the value of the function $6 + x - 2x^2$ be denoted by y, i.e. $y = 6 + x - 2x^2$.

The following table shows corresponding values of x and y.

x	−3	−2	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y	−15	−4	3	5	6	6	5	0	−9

The graph is a parabola cutting the x-axis where $x = -1{\cdot}5$ and where $x = 2$.

$x = -1{\cdot}5$ and $x = 2$ are therefore solutions of the equation $6 + x - 2x^2 = 0$.

The graph rises to its highest point at $x = \frac{1}{4}$, the maximum value of the function being $6\frac{1}{8}$.

The equation $6 + x - 2x^2 = k$ has two solutions when k is less than $6\frac{1}{8}$.

When $k = 6\frac{1}{8}$ the two roots of the equation are equal, and when k is greater than $6\frac{1}{8}$ the roots are imaginary.

The graph of $y = ax^2 + bx + c$ is a parabola in all cases.

When a is positive, as in Case 1, the parabola has a minimum turning point.

When a is negative, as in Case 2, the parabola has a maximum turning point.

Graphical Solution of Quadratic Equations

Solve graphically the equation $4x^2 - 5x - 8 = 0$.

1st Method. Draw the graph of $y = 4x^2 - 5x - 8$.

x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
y	18	$8\frac{1}{2}$	1	$-4\frac{1}{2}$	-8	$-9\frac{1}{2}$	-9	$-6\frac{1}{2}$	-2	$4\frac{1}{2}$	13

The graph is the dotted curve in the figure on p. 287. The graph cuts the axis of x in two points A and B at which the values of x are -0.92 and 2.17.

The roots of the equation are therefore -0.92 and 2.17.

2nd Method. When $4x^2 - 5x - 8 = 0$.

$$\therefore\ 4x^2 = 5x + 8$$

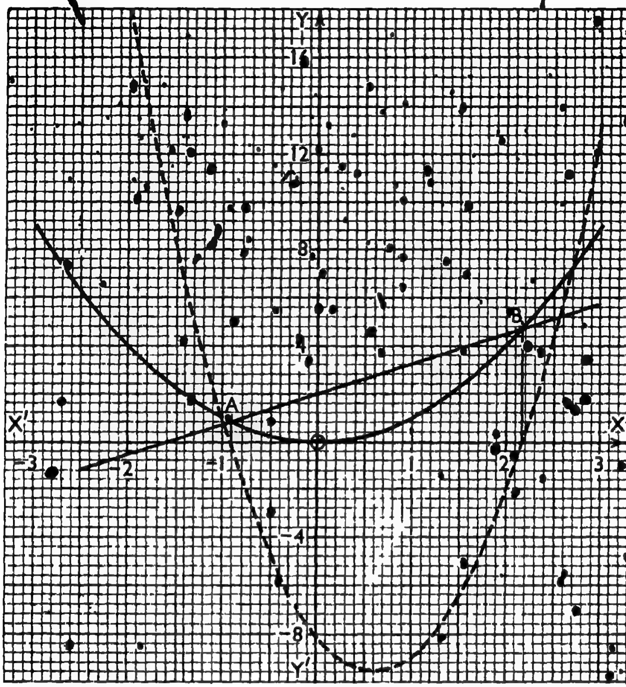
$$\therefore\ x^2 = \frac{5x + 8}{4}$$

The required values of x are therefore those values for which the two functions x^2 and $\frac{5x + 8}{4}$ are equal.

We therefore draw two graphs as explained previously:

(1) $y = x^2$, a parabola, shown as a continuous line in the figure, and

(2) $y = \frac{5x + 8}{4}$, a straight line.



The straight line cuts the parabola in two points A and B. Since the point A lies on both graphs, its co-ordinates satisfy both equations. Let the co-ordinates of A be (x_A, y_A).

$$\therefore\ y_A = x_A^2 \text{ and } y_A = \frac{5x_A + 8}{4}$$

$$\therefore\ x_A^2 = \frac{5x_A + 8}{4}$$

$$\therefore\ 4x_A^2 - 5x_A - 8 = 0$$

$\therefore\ x_A$ is a root of the equation $4x^2 - 5x - 8 = 0$

Similarly, x_B is a root of the equation $4x^2 - 5x - 8 = 0$.

From the graph it is seen that x_A and x_B have the values found by the first method, i.e. $-0{\cdot}92$ and $2{\cdot}17$.

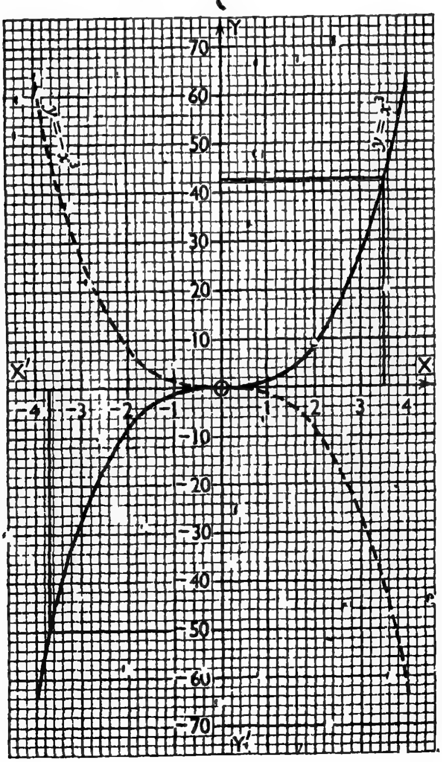
Curve of Cubes

$$y = x^3$$

Corresponding values of x and y are shown in the following table:

-4		-2	-1	0	1	2	3	4
-64	-27	-8	-1	0	1	8	27	64

Since $y = x^3$, it is clear that if x is positive y must be positive, and if x is negative, so also is y.



x and y must therefore always have the same signs, and hence the graph must lie completely in the 1st and 3rd quadrants.

The graph may be used to find approximate values of cubes and approximate values of cube roots.

Thus, from the graph

$$(3\cdot5)^3 = 42\cdot8 \text{ approximately}$$
$$\text{and } \sqrt[3]{-50} = -3\cdot7 \text{ approximately}$$

When $y = -x^3$,

y is negative when x is positive, and positive when x is negative. The graph therefore lies in the 2nd and 4th quadrants, and is shown as a dotted line in the diagram.

Graphical Solution of Cubic Equations of the Type

$$x^3 + ax + b = 0$$

Solve graphically the equation

$$x^3 - 8x + 7 = 0$$

1st Method. Draw the graph of $y = x^3 - 8x + 7$.

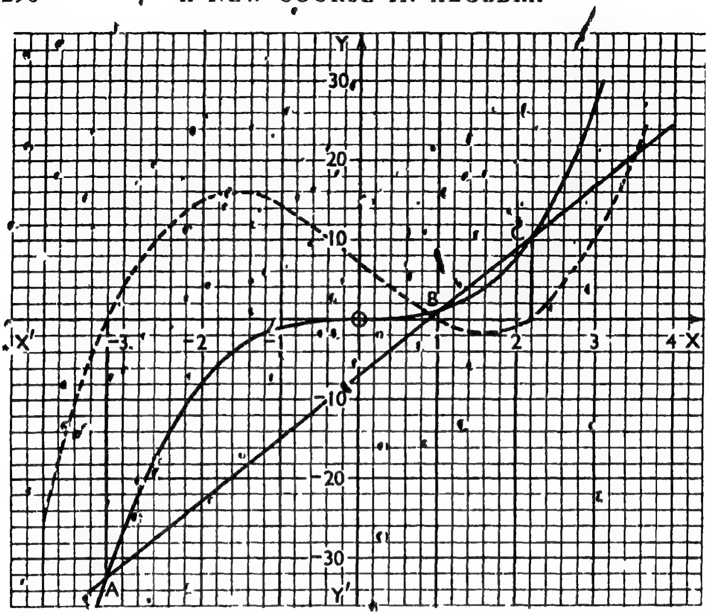
x	-4	-3	-2	$-1\frac{1}{2}$	-1	0	1	$1\frac{1}{2}$	2	3
x^3	-64	-27	-8	$-3\frac{3}{8}$	-1	0	1	$3\frac{3}{8}$	8	27
$-8x$	32	24	16	12	8	0	-8	-12	-16	-24
$+7$	7	7	7	7	7	7	7	7	7	7
y	-25	4	15	$15\frac{5}{8}$	14	7	0	$-1\frac{5}{8}$	-1	10

The graph is the dotted curve in the figure on p. 290.

The graph cuts the axis of x in three points. At each of these points the value of y is zero, i.e.

$$x^3 - 8x + 7 = 0$$

The roots are $-3\cdot2$, 1, $2\cdot2$.



2nd Method. When $x^3 - 8x + 7 = 0$.

$$\therefore x^3 = 8x - 7$$

We draw two graphs:

$$(1)\ y = x^3$$
$$(2)\ y = 8x - 7$$

both of which are shown as continuous lines in the figure.

These graphs intersect at three points A, B, C.

Since A lies on both graphs,

$$y_A = x_A^3$$

and
$$y_A = 8x_A - 7$$
$$\therefore x_A^3 = 8x_A - 7$$
$$\therefore x_A^3 - 8x_A + 7 = 0$$

i.e. x_A is a root of the equation

$$x^3 - 8x + 7 = 0$$

Similarly, x_B and x_C are roots of the equation

$$x^3 - 8x + 7 = 0$$

From the graph, the roots are $-3 \cdot 2$, 1, $2 \cdot 2$ as before.

It is evident from the shape of the curve $y = x^3$ that any straight line drawn on the plane of the axes must cut the curve at least once, i.e. the line $y = ax + b$, whatever be the values of a and b, must cut the curve $y = x^3$ in at least one point.

It follows that every cubic equation of the form $x^3 = ax + b$ has at least one real solution.

The Function $\frac{m}{x}$

(1) *m positive.* Let $y = \frac{m}{x}$. When x is positive y is positive, and when x is negative y is negative.

It follows that the graph of the function lies completely in the first and third quadrants.

When x is small and positive, y is large and positive, and we can make y as large as we please simply by giving x smaller and smaller values.

The graph thus approaches the y axis as x tends to zero.

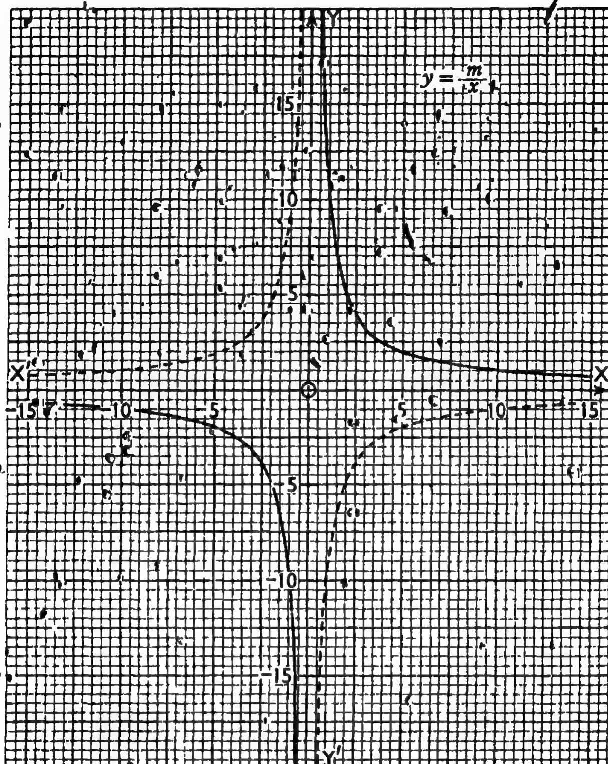
When x is large and positive, y is small and positive, and we can make y as small as we please simply by giving x larger and larger values.

The graph thus approaches the x axis as x tends to infinity.

We say that the x and y axes are *asymptotes* to the graph.

In the third quadrant x is negative and the graph has a similar shape to that in the first quadrant.

(2) *m negative.* When x is positive y is negative, and when x is negative y is positive.



It follows that the graph lies entirely in the second and fourth quadrants and has the same shape as before.

The diagram shows the graph for $m = 10$, i.e.

$$y = \frac{10}{x}$$

drawn as a continuous line and the graph for

$$m = -10, \text{ i.e. } y = \frac{-10}{x}$$

drawn as a dotted line.

Exercises 82

Draw the graphs of the following functions and find in each case the maximum or minimum value of the function.

1. $x^2 - x - 2$
2. $1 + x - 2x^2$
3. $2x^2 + x - 6$
4. $2 - x - 3x^2$

5. Draw the graph of $y = 5 - 2x - 3x^2$. For what range of values of x is $5 - 2x - 3x^2$ positive? What is the maximum value of $5 - 2x - 3x^2$?

6. Draw the graph of $y = 4x^2 - 5x - 6$. For what range of values of x is $4x^2 - 5x - 6$ negative? What is the minimum value of the function?

7. Draw the graph of $y = 2x^2 - x - 2$ and obtain from the graph solutions of the equations:

(i) $2x^2 - x - 3 = 0$ (ii) $4x^2 - 2x - 3 = 0$

8. Draw the graph of $y = 2 + 5x - 3x^2$ and obtain from the graph solutions of the equations:

(i) $4 + 5x - 3x^2 = 0$ (ii) $3x^2 - 5x = 3$

Find in the form $ax^2 + bx + c = 0$ the equations which have as their solutions the values of x at the points of intersection of the following graphs:

9. $y = x^2$; $y = x + 2$
10. $y = x^2$; $y = 2x - 1$
11. $y = x^2$; $y = \frac{1 - x}{2}$
12. $y = x^2$; $y = \frac{4}{3}(x + 1)$

13. Draw the graph of $y = x^2$ and that of $y = 2x + 3$ and use them to solve the equation $x^2 - 2x = 3$.

14. Draw the graph of $y = x^2$. What other graph must be drawn in order to solve the equation $2x^2 - 5x + 2 = 0$. What are the solutions?

15. Draw the graph of $y = x^2$. Draw the straight line through the points on the graph at which x has the values -1 and 2. What is the equation to this straight line, and what equation has for its roots the values of x where the line meets the parabola?

16. Draw the graph of $y = x^3 - 3x + 2$. What are the values of x which make $x^3 - 3x + 2 = 0$?

Use the graph to solve the equations:

(i) $x^3 - 3x + 1 = 0$ (ii) $x^3 = 3(x - 1)$

17. Draw the graph of $y = \frac{1}{3}(x^3 - 5x - 4)$ for values of x between -3 and $+3$ and find from the graph:

(i) the maximum and minimum values of the function $\frac{1}{3}(x^3 - 5x - 4)$;

(ii) the values of x for which $x^3 - 5x - 4$ is negative;

(iii) the solutions of the equations—

(*a*) $x^3 - 5x - 4 = 0$ (*b*) $x^3 - 5x + 2 = 0$

On the same graph draw the graph of $2(x - y) = 1$ and find the values of x at the points of intersection of the graphs. Of what cubic equation are these values of x the roots?

18. On the same diagram draw the graphs of $y = x^3$ and $25x - 6y + 15 = 0$.

Read the values of x at the points of intersection and find the equation of which these values are the roots.

For what values of x is $6x^3 - 25x - 15$ negative?

19. On the same diagram draw the graphs of $y = x^3$ and $y = 6x - 3$ and find the roots of the equation

$$x^3 - 6x + 3 = 0$$

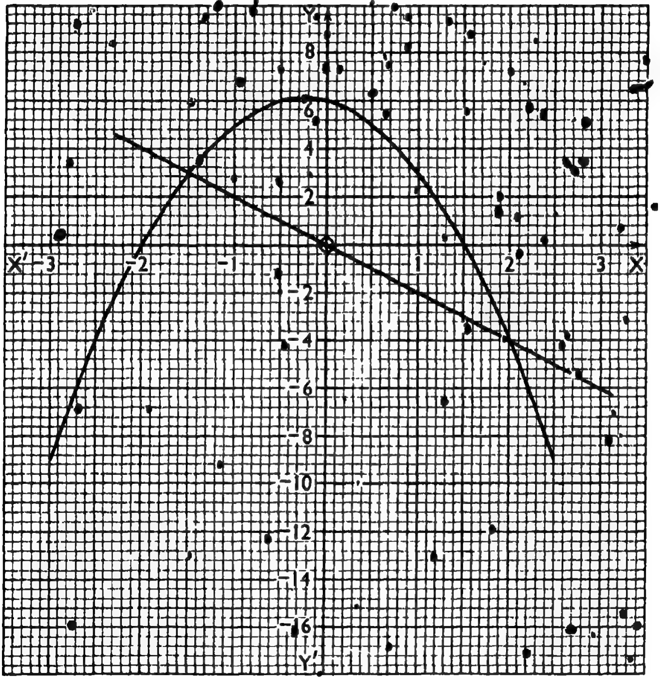
Draw on the same diagram the graph of $5x + y + 6 = 0$. Read the value of x at the point of intersection of this graph with the graph of $y = x^3$ and find the cubic equation of which this is a root.

20. Draw the graph of $y = x(x - 1)(x + 2)$.

(i) For what range of values of k does the equation $x(x - 1)(x + 2) = k$ have three solutions?

(ii) For what positive range of values of x is $x(x - 1)(x + 2)$ negative?

(iii) Solve the equation $x(x - 1)(x + 2) = 4$.



21. The graph in the diagram above is that of

$$y = (3 - 2x)(x + 2)$$

Find from the graph:

(i) the maximum value of the function $(3 - 2x)(x + 2)$;

(ii) the range of values of x for which the function $(3 - 2x)(x + 2)$ is positive;

(iii) the roots of the equation $2x^2 + x - 6 = 0$;

(iv) the roots of the equation $(3 - 2x)(x + 2) + 5 = 0$;

(v) the equation to the straight-line graph;

(vi) the values of x at the points of intersection of the graphs;

(vii) the equation which has these values of x as its roots.

22. The diagram on p. 297 shows the graph of

$$y = x(x - 2)(2x + 3)$$

Find from the graph:

(i) the maximum and minimum values of the function $x(x - 2)(2x + 3)$.

(ii) for what values of x is the function $x(x - 2)(2x + 3)$ positive.

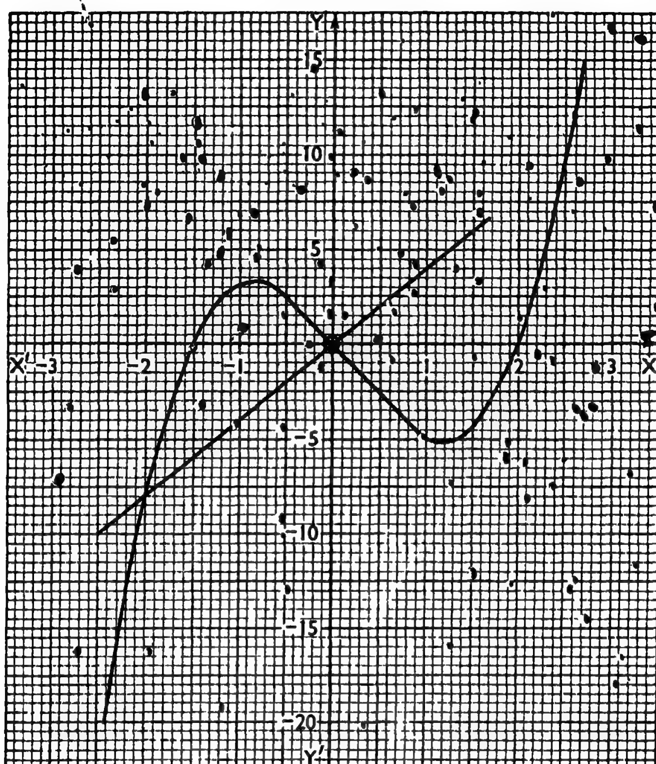
(iii) the roots of the equation $x(x - 2)(2x + 3) = 2$.

(iv) the range of values of k for which the equation $x(x - 2)(2x + 3) = k$ has three roots.

(v) the equation to the straight line joining the points $(-2, -8)$ and $(0, 0)$ on the curve.

(vi) the co-ordinates of the third point in which this line cuts the graph. Check this result algebraically.

23. Draw on the same diagram and to the same scale the graphs of $y = \dfrac{4}{x}$ and $y = \frac{1}{2}(6 - x)$, from $x = +1$ to $x = +5$. Read off the values of x at the points where the graphs intersect and write down an equation of which these values are the roots. Verify the results by solving the equation.



24. Draw between $x = 1$ and $x = 4$, plotting points at half unit intervals, the graphs of $xy = 2$ and $y + 1 = (x - 3)^2$, using the same axes and scales for both graphs. What equation in x is satisfied by the value of x at the points of intersection of these graphs? Read from the graph the root of this equation which lies between $x = 1$ and $x = 4$.

Paper 31

1. (i) Find $x : y$ if $8x^2 - 6xy - 27y^2 = 0$.

(ii) The ratio $2x - y : x - y = 2 : 3$. Find the ratio $3x - y : x + 3y$.

2. (i) Solve: $\dfrac{1}{x-5} - \dfrac{1}{x-2} = \dfrac{2}{3(x-4)}$;

(ii) Simplify: $9^{\frac{1}{2}} \times 27^{-\frac{2}{3}} \times (\frac{1}{81})^{-1}$

3. A man rows $2\frac{3}{4}$ miles down a river, and then back again in 1 hr. 36 min. If the speed of the current is $1\frac{1}{2}$ m.p.h., find the man's rate of rowing in still water.

4. Solve:

$$\text{(i)}\quad \frac{x-p}{x+p} = \frac{p-q}{p+q}$$

$$\text{(ii)}\quad \begin{aligned} a(x-1) &= 1-y \\ x-1 &= b(y-1) \end{aligned}$$

5. If $u^2 = k(p+q)\left(\dfrac{2}{r} - \dfrac{1}{a}\right)$, express r in terms of the other letters.

6. If $\dfrac{a-c}{b-d} = \dfrac{2a+3c}{2b+3d}$, prove that a, b, c, d are in proportion.

Paper 32

1. (i) Simplify: $\dfrac{a - \dfrac{b^2}{a}}{\dfrac{a^3}{b^2} - \dfrac{b^2}{a}} \times \left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)$.

(ii) Evaluate: $\left(\dfrac{x^{-1}y}{x^2y^{-2}}\right)^{-2} \times \left(\dfrac{x^{-3}y^{-3}}{x^{-2}y^2}\right)^{-1}$

when $x = 2$, $y = 9$.

2. Solve:

(i) $\dfrac{x}{x+1} + \dfrac{x+4}{x+5} = \dfrac{2x+6}{x+4}$;

(ii) $x^2 + 6x + 2 = 0$, correct to two decimal places.

3. A man motors a distance of 80 miles. If he had travelled at an average speed 5 m.p.h. less he would have taken 32 min. longer on the journey. What was his average speed?

4. If the coefficient of x^2 in the product of $2x^2 - 3x + 4$ and $x^2 + kx - 3$ is 10, find k. Find also the coefficient of x in the product when k has this value.

5. Solve:

(i) $\dfrac{1}{x+a} + \dfrac{1}{x+b} = \dfrac{2}{x}$

(ii) $\dfrac{2a}{x} - \dfrac{b}{y} = \dfrac{1}{6} = \dfrac{3a}{2x} - \dfrac{2b}{3y}$

6. If $T^2 = 4\pi^2\left(\dfrac{b^2+k^2}{hg}\right)$, find an expression for k in terms of the other letters.

Paper 33

1. (i) Simplify:

$$\frac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{c}} \times \frac{\dfrac{1}{a+b}-\dfrac{1}{c}}{\dfrac{1}{a-c}+\dfrac{1}{b}}$$

(ii) Find the value of n if (*a*) $2^n = 32$, (*b*) $8^n = 2$, (*c*) $2^n = \frac{1}{16}$, (*d*) $4^{-n} = 2$, (*e*) $16^n = \sqrt[3]{2^2}$

2. Solve the equations:

(i) $\dfrac{3}{x-1} - \dfrac{12}{x-2} + \dfrac{20}{x-7} = 0$;

(ii) $x^2 + 3x - 2 = 0$, correct to two decimal places.

3. A man bought a number of articles for which he paid £54. If each article had cost 3*s*. less he could have bought 4 more articles for the same money. How many articles did he buy?

4. (i) Simplify $\sqrt{50} + 3\sqrt{2} - \sqrt{32}$.

(ii) Evaluate $\frac{4}{\sqrt{6}}$ correct to two decimal places.

(iii) Expand $(3\sqrt{5} + 2)^2$.

5. Solve the equations:

$$\text{(i)}\quad \frac{a}{x+a} + \frac{b}{x+b} = 1.$$

$$\text{(ii)}\quad \frac{x-m}{n} = y - m = \frac{x-n}{m}.$$

6. Change the subject of the following formula to *h*.

$$A = \pi r\sqrt{(h^2 + r^2)} + \pi r^2$$

Paper 34

1. If $x^2 + 4x - 2$ is a factor of $x^4 + 5x^3 - 10x + 4$, find the other factors.

2. Solve:

$$\text{(i)}\quad \frac{3x-5}{x+2} - \frac{x+2}{3x-5} = 3\tfrac{3}{4};$$

(ii) $2x^2 + 4x - 3 = 0$, correct to two decimal places.

3. A journey of 400 miles is accomplished in a total time of 8 hr. 20 min. If the average speed on the first half of the journey is 20 m.p.h. less than on the second half, find the average speed for each half.

4. (i) If y varies directly as x^2, and $y = 16$ when $x = 8$, find the value of x when $y = 9$.

(ii) If y varies inversely as x^3, and $y = \frac{1}{32}$ when $x = -4$, find the value of y when $x = \frac{1}{2}$.

5. Solve:

(i) $\dfrac{1}{x-a} - \dfrac{1}{x} = \dfrac{1}{x+a} - \dfrac{1}{x+2a}$

(ii) $\dfrac{x}{a} = \dfrac{a}{x} + \dfrac{2}{x} + \dfrac{1}{ax}$

6. Make u the subject of the formula

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}$$

Paper 35

1. Solve:

(i) $\dfrac{x+2}{4(2x-3)} - \dfrac{x-2}{5(x-1)} = \dfrac{1}{2x^2 - 5x + 3}$;

(ii) $2x^2 + 5x + 1 = 0$, correct to two decimal places.

2. A bookseller buys a number of books at a price in pence which is 2 greater than the number of books bought. If the books cost him £4, how many did he buy?

3. The weight of a metal sphere varies jointly as the cube of the radius and the density of the metal. If the weight of a metal sphere of radius 10 in. is 300 lb., find the weight of a second metal sphere of radius 5 in., if the metal in the second sphere is four times as dense as that of the first.

4. (i) What number must be subtracted from 8, 11, 20, 29 so that the resulting numbers will be in proportion?

(ii) Solve
$$a(x+y) = b(x-y)$$
$$a(x-y) + b(x+y) = a^2 + b^2$$

5. (i) If $\dfrac{(a-b)(a+b)}{b(a-c)} = \dfrac{b}{c}$, prove that b is a mean proportional between a and c.

(ii) Solve $\dfrac{2x^2 - 3x + 5}{3x - 5} = \dfrac{x^2 + 4x - 3}{3 - 4x}$

6. If $n = \dfrac{1}{2rl}\sqrt{\dfrac{Tg}{\pi d}}$, change the subject to T and find the value of T, if $r = 0{\cdot}05$, $l = 1{\cdot}2$, $n = 700$, $d = 9$, $\pi = 3\frac{1}{7}$, $g = 32$.

EXAMINATION PAPERS

Paper 1

1. (i) Show that, if $x = 1 + \sqrt{2}$ and $y = 1 + \sqrt{3}$, then $3x^2 - 6x + 3 = 2y^2 - 4y + 2$.

(ii) Evaluate $\dfrac{xy}{x^2 + y^2}$ and $\dfrac{\sqrt{2xy}}{x - y}$ when $x : y = 2 : 3$.

[P.]

2. Solve the equation $2x^2 - 5x - 9 = 0$, giving your answers correct to two decimal places. [N.]

3. The distance from my house to school is $1\frac{1}{2}$ miles. If I walk the first $\frac{1}{2}$ mile and run the remainder the journey takes me 20 min. If I walk $\frac{4}{5}$ mile and run the rest of the way it takes me 23 min.

If I walk 1 mile in x min. and run 1 mile in y min., calculate the value of y. Hence find my running speed in miles per hour. [C.]

4. (i) Solve the equation $\dfrac{3x}{4} - \frac{1}{2}(x - 1) = 0$.

(ii) Express as a single fraction

$$\frac{x}{x - 3} - \frac{x + 3}{3x}$$

(iii) If $V = 13\left(\dfrac{M}{n} - n\right)$, express M in terms of V and n.

[B.]

5. (i) Express a in terms of b, given that $b = 1 + \dfrac{4}{a}$.

Given also that $c = 1 - \dfrac{3}{b}$, show that

$$a + b + c = \frac{b^3 + 3}{b(b - 1)}$$

(ii) Express in algebraic symbols the following statement, and verify that it is true:

The average of the squares of two numbers together with the product of the numbers is equal to twice the square of the average of the numbers. [S.]

Paper 2

1. Two numbers x and y are connected by the formula $y = \dfrac{16x - 45}{3x - 8}$; find the two values of x for which $y = x$.

If a is the smaller and b the larger of these values, express $\dfrac{y - a}{y - b}$ in terms of x. [B.]

2. Using the same axes and the same scales, draw the graphs of $y - 2 = \frac{1}{2}(x - 1)(4 - x)$ and $3y = 2x$, from $x = -1$ to $x = 6$. From your graphs find the range of values of x for which $2 + \frac{1}{2}(x - 1)(4 - x) > \frac{2}{3}x$. [N.]

3. (i) Divide $6x^3 - 5x^2 - 8x + 3$ by $2x - 3$.

(ii) Factorise: (1) $2x^2 - x - 15$; (2) $xy - 3x + 2y - 6$.

(iii) Given that $y = ax + bx^2$, when a, b are constants, and that $y = 4$ when $x = 2$ or 4, find y when $x = 6$. [W.]

4. Simplify $(a - b)^2 + 2b(a + b) - (a^2 + b^2)$, and find the value of the given expression and of your answer when $a = 1$, $b = 2$. [C.]

5. A man bought 6 standard roses and 18 bush roses for £8 2*s*. Another man bought, at the same prices, 14 standard roses and 27 bush roses for £15 18*s*. How much did a standard rose cost? [L.]

Paper 3.

1. (*a*) Simplify:

$$(3p - 2q + r) - (5p + 2q - 5r) - 4(p + q)$$

(*b*) If $y = 3 - 2x$, express in its simplest form, in terms of x, $x^2 - 2xy + y^2$.

(*c*) Simplify $\frac{x}{1\!\!\!\!\!\;\;} - \frac{x-y}{3} + \frac{x+y}{2}$.

(*d*) Find the value of $ax^2 + bx + c$ if $a = 2$, $b = 3$, $c = 1$, $x = -4$. [N.]

2. (*a*) Given that $x = \frac{1+a}{2a}$ and $y = \frac{1+a^2}{1-a^2}$, prove that $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

(*b*) Multiply $x^2 + 2x - 8$ by $x - p$. Find the value of p which makes the coefficient of x in the product zero.

Write down the product when p has this value.

(*c*) Find the quotient and remainder when $2x^3 + 5x^2 - 4$ is divided by $2x - 1$. [S.]

3. (i) Solve the equation:

$$\frac{5x}{2x+3} = \frac{x}{6} + 1$$

(ii) A boy usually cycles to school, a distance of 3 miles, at 12 m.p.h. One day he had a puncture after cycling part of the way and walked the remainder pushing his cycle at 3 m.p.h. In consequence the journey took 10 min. longer than usual. How far from school did the puncture occur? [W.]

4. (i) A toy which costs s shillings is sold at an increase of $p\%$ on this price. Given that 1 rupee is worth r shillings, express in terms of s, p and r the selling price in rupees.

(ii) Express h in terms of π, S and r from the formula $S = \pi r(2h + r)$. [C.]

5. (i) Solve the equations:

$$3x + 4y + 2 = 0$$
$$2x - 2y + 3 = 0$$

(ii) An express train travelling at 75 m.p.h. is 549 ft. long. If it takes 12 sec. to pass completely a goods train travelling in the opposite direction at 15 m.p.h., find the length of the goods train. [B.]

Paper 4

1. Simplify:

(i) $\dfrac{1}{t-2} - \dfrac{1}{2t+1} + \dfrac{1}{2(t-2)}$

(ii) $\dfrac{3a+1}{a^2+2a} \times \dfrac{a+2}{3a^2-7a+2}$ [C.]

2. (i) Solve the equation $3x^2 - 15x + 17 = 0$, giving the answers correct to two decimal places.

(ii) A second-hand car, bought by a dealer for £x, was sold by him for £144. If the dealer made a gain of x% by this sale, find x. [B.]

3. (i) A grocer buys oranges at x for a shilling and sells them at y pence each. What is his profit per cent on his outlay?

(ii) If $T = 2\pi\sqrt{\left\{\dfrac{h^2+k^2}{gh}\right\}}$, find an expression for k in terms of the other letters. [W.]

4. (i) Find the value of x and the value of y which satisfy the equations:

$$2x - y = \tfrac{3}{7}(2x + y) = 6x - 7y + 4$$

(ii) Given that $\dfrac{5a+4b}{a+3b} = \dfrac{3}{2}$, find the value of $\dfrac{a}{b}$. [L.]

5. One number exceeds another number by three. Three times the smaller exceeds twice the greater by one. Find the numbers. [N.]

Paper 5

1. (i) Multiply $5x^2 - 7x - 3$ by $x^2 + x - 4$.

(ii) Divide $6x^4 - 3x^3 - 2x^2 + 7x - 28$ by $3x^2 - 7$. [L.]

2. (i) Simplify $2(3a^3 - 4b^3) - 3(a^3 - b^3)$ and find its value when $a = -1$, $b = -2$.

(ii) Solve the equation $\frac{3x-4}{6} - \frac{x-1}{4} = \frac{1}{3}$.

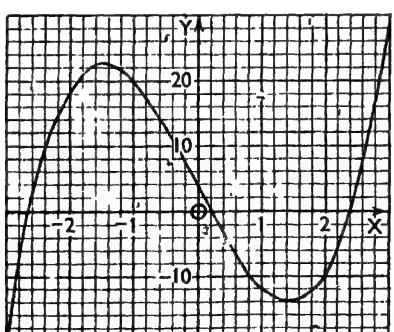
(iii) When an expression E is divided by $x + 2$, the quotient is $2x - 3$ and the remainder is 9. Find E and express your result in its simplest form. [W.]

3. (i) If x and y are given in terms of a and b by the equations $2a + b = x$, $2x + b = y$, express y in terms of a and b. If 6 is added to the number a, what is added to y?

(ii) Solve the equations:

$$x - 4y = 1$$
$$3x + 2y = 24$$ [R.]

4. The sum of three numbers is $3\frac{13}{24}$. Two of them differ by $\frac{1}{12}$ and the third number is half as much again as the sum of these two. Find the numbers. [O.]



5. The graph of a function of x is shown above for values of x from -3 to $+3$.

Reading values of x to the second decimal place and values of the function to the first decimal place, find from the graph:

(i) the minimum turning value of the function;
(ii) the value of the function when $x = -1{\cdot}6$;
(iii) the values of x for which the function is zero;
(iv) the range of positive values of x for which the function is negative;

It is known that the function is of the form $3x^3 + ax + b$, where a and b are constants. Find the values of a and b.
[S.]

Paper 6

1. (i) Simplify $(3x + y)(x - 12y) - (2x + y)(x - 16y)$ and hence show that this expression is a perfect square.

(ii) If $S = \dfrac{bt - a}{t - b}$, find the value of t in terms of S, a, b, and check your result when $t = 3$, $b = 2$, $a = -1$.
[D.]

2. (i) Factorise, expressing each factor in its simplest form:

$$(a)\quad ax - bx - ax^2 + bx^2$$
$$(b)\quad (a + b)^2 - (c - d)^2$$

(ii) The external dimensions of a closed rectangular box are x, $(x + 1)$ and $(x + 2)$ in. respectively. Write down and simplify an expression for the total external area of the box.
[L.]

3. (i) Solve the simultaneous equations:

$$2x + 3y + 10 = 0$$
$$3x - 4y - 53 = 0$$

(ii) Solve the equation $2z^2 - 7z + 5{\cdot}2 = 0$, giving the roots correct to two decimal places.
[O.]

4. Two positive numbers differ by 18. Twice their product exceeds 3 times their sum by 14. Find the numbers. [P.]

5. The prices of the seats in a cinema are 3*s.* and 2*s.* 3*d.* When every seat is occupied the drawings amount to £135. The seating is rearranged, 20% of the dearer seats being transferred to the cheaper class, and the prices of the seats being increased to 3*s.* 6*d.* and 2*s.* 6*d.*, The receipts for a full house now amount to £149 10*s.* How many seats were there at each of the original prices? [S.]

Paper 7

1. (i) If $x = \dfrac{3y + 7}{2(y - 5)}$, show that $y = \dfrac{10x + 7}{2x - 3}$.

(ii) If x and y in (i) are equal, find their possible values. [B.]

2. (i) Factorise $12x^3 - 2x^2y - 4xy^2$

(ii) Solve the equations:

(*a*) $\frac{2}{3}x(3x + 1) - \frac{1}{2}(2x - 1)^2 = 1$;

(*b*) $3x^2 - 5x - 4 = 0$, giving the roots correct to two decimal places. [S.]

3. Factorise:

(i) $4x^2 - 12xy + 9y^2$

(ii) $6a^2x + 4ax - 2x$

(iii) $9a^2 - (3a - 2b)^2$ [C.]

4. Simplify:

$$\frac{(a^2 - b^2)^2 - (a^2 + b^2)^2}{(a^2 - b^2)^2 - (a + b)^4}$$

and evaluate it when $a/b = 3$.

Show that the value is the same when $a/b = \frac{1}{3}$. [P.]

5. The distance from a man's house to the nearest village is $2\frac{1}{2}$ miles by road and 2 miles by a footpath across fields. The time taken by the man to reach the village cycling along

the road is 25 min. less than the time he takes to walk across the fields. If he cycles 7 m.p.h. faster than he walks, find his rate of walking. [W.]

Paper 8

1. (i) Factorise $4a^2 - 81b^2$ and $3x^2 - 7x - 6$.

(ii) Simplify $\dfrac{5}{2x - 3} - \dfrac{4x + 5}{2x^2 + x - 3}$.

(iii) Simplify $\dfrac{4x + 5}{2x^2 - x - 3} \div \dfrac{x}{x + 1}$. [B.]

2. (i) If $E = 9kn/(3k + n)$ and $\sigma = \frac{1}{2}(3k - 2n)/(3k + n)$, express k in terms of E and σ, and show that if $\sigma = \frac{1}{3}$, $k = E = 8n/3$.

(ii) Express $x^3 - y^3$ in terms of $u = x + y$ and $v = x - y$. Hence find the value of $52^3 - 48^3$. [P.]

3. A man borrowed £P. For the first year he had to pay interest at 5%, and in settlement of this, and to reduce the debt, he paid £N at the end of the first year. Find how much was still owing.

For the second year interest was charged at 10% on the amount still owing, and by paying another £N at the end of the second year the man paid this interest and the remaining part of the debt. Express N as a fraction of P. [O.]

4. (a) If $x : y = 3 : 2$, find $\dfrac{x^3 + y^3}{x + y}$ in terms of: (i) x only; (ii) y only.

(b) If $\dfrac{p^2 + q^2}{p^2 - pq - q^2} = 2$, find the values of $\dfrac{p}{q}$. [N.]

5. (i) Solve the equations:

(a) $\frac{1}{2}(2x - 1)(3x - 1) - \frac{2}{3}(x - 4)(3x + 2) = x^2$;

(b) $2x^2 - 7x - 1 = 0$, expressing the roots correct to two decimal places.

(ii) From the relation $nE = I(r + Rn)$, derive a formula for n in terms of E, I, r and R. [S.]

Paper 9

1. (i) Find the value of $3x^2 - \frac{y}{z}$ when $x = -\frac{1}{3}$, $y = \frac{1}{6}$, $z = -\frac{1}{2}$.

(ii) Simplify $(2m + 3n)(m - 2n) + (3m + 2n)(2m - n)$ and express your answer in factors.

(iii) Given that $2p - 3$ is a factor of $6p^3 + p^2 - 19p + 6$, find the remaining factors. [W.]

2. If $x = \frac{y}{1 - y}$ and $y = \frac{z}{1 - z}$, prove that:

(i) $x = \frac{z}{1 - 2z}$ (ii) $\frac{x + 2y}{2x - y} + 5z = 3$ [O.]

3. The cost of making x articles is £$(a + k\sqrt{x})$, when a and k are constant in value. If the cost of making 25 articles is £20, and the average cost of one article when 100 are made is 6 shillings, find the values of a and k, and the cost of making 16 articles. [N.]

4. (*a*) Factorise $6ax^2 - 4axy - 9bx + 6by$.

(*b*) Solve the equations:

(i) $\frac{x - 2}{3} - \frac{3y - 1}{2} + 1 = 0;$

$x + 6y = 1;$

(ii) $2x^2 - 6x + 3 = 0$, giving the roots correct to two decimal places. [S.]

5. A man is 6 times as old as his son. In 4 years the father will be 4 times as old as his son. What are their present ages? [P.]

Paper 10

1. During a sale the price of cloth is reduced by 4*s.* 6*d.* a yard, and, in consequence, a customer bought 10 more yards for £27 than were previously obtainable. Find the number of yards bought by the customer during the sale for the same sum. [C.]

2. Tabulate to two decimal places the values of $y = \frac{1}{x}$ and of $y = -x^2 - 1$ for the values $\pm 0{\cdot}5$, $\pm 0{\cdot}6$, $\pm 0{\cdot}7$, ± 1, ± 2 of x. Draw on the same diagram the graphs of these functions from $x = -2$ to $x = +2$, taking 1 in. as unity on each axis. From your figure obtain an approximate value of the real root of the equation $x^3 + x + 1 = 0$. [P.]

3. The annual incomes of two workers A and B are in the ratio 6 : 5 and their expenditures are in the ratio 8 : 7. If A saves £80 per annum and B saves £30 per annum, calculate how much each earns per annum. [L.]

4. (i) Solve the equation:

$$(x - 2)(x - 3) - (x - 1)(x - 2) = 3(x - 3)$$

(ii) Three dozen protractors cost £2. The protractors were of two kinds, one kind costing 10*d.* each, the other kind 1*s.* 4*d.* each. How many were there of each kind? [S.]

5. (i) If $S = \pi r^2 h$, express r in terms of the other letters.

(ii) A car runs 200 miles per week. It uses a gallon of petrol every n miles and a pint of oil every k miles. Petrol costs x shillings per gallon and oil y shillings per pint. Petrol and oil together account for $p\%$ of the total weekly expenditure on the car. Find an expression in pounds for the remaining weekly expenditure. [O.]

SQUARE ROOTS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	1000	1005	1010	1015	1020	1025	1030	1034	1039	1044	0	1	1	2	2	3	3	4	4
	3162	3178	3194	3209	3225	3240	3256	3271	3286	3302	2	3	5	6	8	9	11	12	14
11	1049	1054	1058	1063	1068	1072	1077	1082	1086	1091	0	1	1	2	2	3	3	4	4
	3317	3332	3347	3362	3376	3391	3406	3421	3435	3450	1	3	4	6	7	9	10	12	13
12	1095	1100	1105	1109	1114	1118	1122	1127	1131	1136	0	1	1	2	2	3	3	4	4
	3464	3479	3493	3507	3521	3536	3550	3564	3578	3592	1	3	4	6	7	8	10	11	13
13	1140	1145	1149	1153	1158	1162	1166	1170	1175	1179	0	1	1	2	2	3	3	3	4
	3606	3619	3633	3647	3661	3674	3688	3701	3715	3728	1	3	4	5	7	8	10	11	12
14	1183	1187	1192	1196	1200	1204	1208	1212	1217	1221	0	1	1	2	2	3	3	3	4
	3742	3755	3768	3782	3795	3808	3821	3834	3847	3860	1	3	4	5	7	8	9	11	12
15	1225	1229	1233	1237	1241	1245	1249	1253	1257	1261	0	1	1	2	2	2	3	3	4
	3873	3886	3899	3912	3924	3937	3950	3962	3975	3987	1	3	4	5	6	8	9	10	11
16	1265	1269	1273	1277	1281	1285	1288	1292	1296	1300	0	1	1	2	2	2	3	3	4
	4000	4012	4025	4037	4050	4062	4074	4087	4099	4111	1	2	4	5	6	7	9	10	11
17	1304	1308	1311	1315	1319	1323	1327	1330	1334	1338	0	1	1	2	2	2	3	3	3
	4123	4135	4147	4159	4171	4183	4195	4207	4219	4231	1	2	4	5	6	7	8	10	11
18	1342	1345	1349	1353	1356	1360	1364	1367	1371	1375	0	1	1	1	2	2	3	3	3
	4243	4254	4266	4278	4290	4301	4313	4324	4336	4347	1	2	3	5	6	7	8	9	10
19	1378	1382	1386	1389	1393	1396	1400	1404	1407	1411	0	1	1	1	2	2	3	3	3
	4359	4370	4382	4393	4405	4416	4427	4438	4450	4461	1	2	3	5	6	7	8	9	10
20	1414	1418	1421	1425	1428	1432	1435	1439	1442	1446	0	1	1	1	2	2	2	3	3
	4472	4483	4494	4506	4517	4528	4539	4550	4561	4572	1	2	3	4	5	7	8	9	10
21	1449	1453	1456	1459	1463	1466	1470	1473	1476	1480	0	1	1	1	2	2	2	3	3
	4583	4593	4604	4615	4626	4637	4748	4658	4669	4680	1	2	3	4	5	6	8	9	10
22	1483	1487	1490	1493	1497	1500	1503	1507	1510	1513	0	1	1	1	2	2	2	3	3
	4690	4701	4712	4722	4733	4743	4754	4764	4775	4785	1	2	3	4	5	6	7	8	9
23	1517	1520	1523	1526	1530	1533	1536	1539	1543	1546	0	1	1	1	2	2	2	3	3
	4796	4806	4817	4827	4837	4848	4858	4868	4879	4889	1	2	3	4	5	6	7	8	9
24	1549	1552	1556	1559	1562	1565	1568	1572	1575	1578	0	1	1	1	2	2	2	3	3
	4899	4909	4919	4930	4940	4950	4960	4970	4980	4990	1	2	3	4	5	6	7	8	9
25	1581	1584	1587	1591	1594	1597	1600	1603	1606	1609	0	1	1	1	2	2	2	3	3
	5000	5010	5020	5030	5040	5050	5060	5070	5079	5089	1	2	3	4	5	6	7	8	9
26	1612	1616	1619	1622	1625	1628	1631	1634	1637	1640	0	1	1	1	2	2	2	2	3
	5099	5109	5119	5128	5138	5148	5158	5167	5177	5187	1	2	3	4	5	6	7	8	9
27	1643	1646	1649	1652	1655	1658	1661	1664	1667	1670	0	1	1	1	2	2	2	2	3
	5196	5206	5215	5225	5235	5244	5254	5263	5273	5282	1	2	3	4	5	6	7	8	9
28	1673	1676	1679	1682	1685	1688	1691	1694	1697	1700	0	1	1	1	1	2	2	2	3
	5292	5301	5310	5320	5329	5339	5348	5357	5367	5376	1	2	3	4	5	6	7	7	8
29	1703	1706	1709	1712	1715	1718	1720	1723	1726	1729	0	1	1	1	1	2	2	2	3
	5385	5394	5404	5413	5422	5431	5441	5450	5459	5468	1	2	3	4	5	5	6	7	8
30	1732	1735	1738	1741	1744	1746	1749	1752	1755	1758	0	1	1	1	1	2	2	2	3
	5477	5486	5495	5505	5514	5523	5532	5541	5550	5559	1	2	3	4	4	5	6	7	8
31	1761	1764	1766	1769	1772	1775	1778	1780	1783	1786	0	1	1	1	1	2	2	2	3
	5568	5577	5586	5595	5604	5612	5621	5630	5639	5648	1	2	3	3	4	5	6	7	8
32	1789	1792	1794	1797	1800	1803	1806	1808	1811	1814	0	1	1	1	1	2	2	2	2
	5657	5666	5675	5683	5692	5701	5710	5718	5727	5736	1	2	3	3	4	5	6	7	8

The first significant figure and the position of the decimal point must be determined by inspection.

SQUARE ROOTS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
33	1817	1819	1822	1825	1828	1830	1833	1836	1838	1841	0	1	1	1	1	2	2	2	2
	5745	5753	5762	5771	5779	5788	5797	5805	5814	5822	1	2	3	3	4	5	6	7	8
34	1844	1847	1849	1852	1855	1857	1860	1863	1865	1868	0	1	1	1	1	2	2	2	2
	5831	5840	5848	5857	5865	5874	5882	5891	5899	5908	1	2	3	3	4	5	6	7	8
35	1871	1873	1876	1879	1881	1884	1887	1889	1892	1895	0	1	1	1	1	2	2	2	2
	5916	5925	5933	5941	5950	5958	5967	5975	5983	5992	1	2	2	3	4	5	6	7	8
36	1897	1900	1903	1905	1908	1910	1913	1916	1918	1921	0	1	1	1	1	2	2	2	2
	6000	6008	6017	6025	6033	6042	6050	6058	6066	6075	1	2	2	3	4	5	6	7	7
37	1924	1926	1929	1931	1934	1936	1939	1942	1944	1947	0	1	1	1	1	2	2	2	2
	6083	6091	6099	6107	6116	6124	6132	6140	6148	6156	1	2	2	3	4	5	6	7	7
38	1949	1952	1954	1957	1960	1962	1965	1967	1970	1972	0	1	1	1	1	2	2	2	2
	6164	6173	6181	6189	6197	6205	6213	6221	6229	6237	1	2	2	3	4	5	6	6	7
39	1975	1977	1980	1982	1985	1987	1990	1992	1995	1997	0	1	1	1	1	2	2	2	2
	6245	6253	6261	6269	6277	6285	6293	6301	6309	6317	1	2	2	3	4	5	6	6	7
40	2000	2002	2005	2007	2010	2012	2015	2017	2020	2022	0	0	1	1	1	1	2	2	2
	6325	6332	6340	6348	6356	6364	6372	6380	6387	6395	1	2	2	3	4	5	6	6	7
41	2025	2027	2030	2032	2035	2037	2040	2042	2045	2047	0	0	1	1	1	1	2	2	2
	6403	6411	6419	6427	6434	6442	6450	6458	6465	6473	1	2	2	3	4	5	5	6	7
42	2049	2052	2054	2057	2059	2062	2064	2066	2069	2071	0	0	1	1	1	1	2	2	2
	6481	6488	6496	6504	6512	6519	6527	6535	6542	6550	1	2	2	3	4	5	5	6	7
43	2074	2076	2078	2081	2083	2086	2088	2090	2093	2095	0	0	1	1	1	1	2	2	2
	6557	6565	6573	6580	6588	6595	6603	6611	6618	6626	1	2	2	3	4	5	5	6	7
44	2098	2100	2102	2105	2107	2110	2112	2114	2117	2119	0	0	1	1	1	1	2	2	2
	6633	6641	6648	6656	6663	6671	6678	6686	6693	6701	1	2	2	3	4	4	5	6	7
45	2121	2124	2126	2128	2131	2133	2135	2138	2140	2142	0	0	1	1	1	1	2	2	2
	6708	6716	6723	6731	6738	6745	6753	6760	6768	6775	1	1	2	3	4	4	5	6	7
46	2145	2147	2149	2152	2154	2156	2159	2161	2163	2166	0	0	1	1	1	1	2	2	2
	6782	6790	6797	6804	6812	6819	6826	6834	6841	6848	1	1	2	3	4	4	5	6	7
47	2168	2170	2173	2175	2177	2179	2182	2184	2186	2189	0	0	1	1	1	1	2	2	2
	6856	6863	6870	6877	6885	6892	6899	6907	6914	6921	1	1	2	3	4	4	5	6	7
48	2191	2193	2195	2198	2200	2202	2205	2207	2209	2211	0	0	1	1	1	1	2	2	2
	6928	6935	6943	6950	6957	6964	6971	6979	6986	6993	1	1	2	3	4	4	5	6	6
49	2214	2216	2218	2220	2223	2225	2227	2229	2232	2234	0	0	1	1	1	1	2	2	2
	7000	7007	7014	7021	7029	7036	7043	7050	7057	7064	1	1	2	3	4	4	5	6	6
50	2236	2238	2241	2243	2245	2247	2249	2252	2254	2256	0	0	1	1	1	1	2	2	2
	7071	7078	7085	7092	7099	7106	7113	7120	7127	7134	1	1	2	3	4	4	5	6	6
51	2258	2261	2263	2265	2267	2269	2272	2274	2276	2278	0	0	1	1	1	1	2	2	2
	7141	7148	7155	7162	7169	7176	7183	7190	7197	7204	1	1	2	3	4	4	5	6	6
52	2280	2283	2285	2287	2289	2291	2293	2296	2298	2300	0	0	1	1	1	1	2	2	2
	7211	7218	7225	7232	7239	7246	7253	7259	7266	7273	1	1	2	3	3	4	5	6	6
53	2302	2304	2307	2309	2311	2313	2315	2317	2319	2322	0	0	1	1	1	1	2	2	2
	7280	7287	7294	7301	7308	7314	7321	7328	7335	7342	1	1	2	3	3	4	5	5	6
54	2324	2326	2328	2330	2332	2335	2337	2339	2341	2343	0	0	1	1	1	1	1	2	2
	7348	7355	7362	7369	7376	7382	7389	7396	7403	7409	1	1	2	3	3	4	5	5	6

The first significant figure and the position of the decimal point must be determined by inspection.

SQUARE ROOTS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	2345	2347	2349	2352	2354	2356	2358	2360	2362	2364	0	0	1	1	1	1	1	2	2
	7416	7423	7430	7436	7443	7450	7457	7463	7470	7477	1	1	2	3	3	4	5	5	6
56	2366	2369	2371	2373	2375	2377	2379	2381	2383	2385	0	0	1	1	1	1	1	2	2
	7483	7490	7497	7503	7510	7517	7523	7530	7537	7543	1	1	2	3	3	4	5	5	6
57	2387	2390	2392	2394	2396	2398	2400	2402	2404	2406	0	0	1	1	1	1	1	2	2
	7550	7556	7563	7570	7576	7583	7589	7596	7603	7609	1	1	2	3	3	4	5	5	6
58	2408	2410	2412	2415	2417	2419	2421	2423	2425	2427	0	0	1	1	1	1	1	2	2
	7616	7622	7629	7635	7642	7649	7655	7662	7668	7675	1	1	2	3	3	4	5	5	6
59	2429	2431	2433	2435	2437	2439	2441	2443	2445	2447	0	0	1	1	1	1	1	2	2
	7681	7688	7694	7701	7707	7714	7720	7727	7733	7740	1	1	2	3	3	4	5	5	6
60	2449	2452	2454	2456	2458	2460	2462	2464	2466	2468	0	0	1	1	1	1	1	2	2
	7746	7752	7759	7765	7772	7778	7785	7791	7797	7804	1	1	2	3	3	4	4	5	6
61	2470	2472	2474	2476	2478	2480	2482	2484	2486	2488	0	0	1	1	1	1	1	2	2
	7810	7817	7823	7829	7836	7842	7849	7855	7861	7868	1	1	2	3	3	4	4	5	6
62	2490	2492	2494	2496	2498	2500	2502	2504	2506	2508	0	0	1	1	1	1	1	2	2
	7874	7880	7887	7893	7899	7906	7912	7918	7925	7931	1	1	2	3	3	4	4	5	6
63	2510	2512	2514	2516	2518	2520	2522	2524	2526	2528	0	0	1	1	1	1	1	2	2
	7937	7944	7950	7956	7962	7969	7975	7981	7987	7994	1	1	2	3	3	4	4	5	6
64	2530	2532	2534	2536	2538	2540	2542	2544	2546	2548	0	0	1	1	1	1	1	2	2
	8000	8006	8012	8019	8025	8031	8037	8044	8050	8056	1	1	2	2	3	4	4	5	6
65	2550	2551	2553	2555	2557	2559	2561	2563	2565	2567	0	0	1	1	1	1	1	2	2
	8062	8068	8075	8081	8087	8093	8099	8106	8112	8118	1	1	2	2	3	4	4	5	5
66	2569	2571	2573	2575	2577	2579	2581	2583	2585	2587	0	0	1	1	1	1	1	2	2
	8124	8130	8136	8142	8149	8155	8161	8167	8173	8179	1	1	2	2	3	4	4	5	5
67	2588	2590	2592	2594	2596	2598	2600	2602	2604	2606	0	0	1	1	1	1	1	2	2
	8185	8191	8198	8204	8210	8216	8222	8228	8234	8240	1	1	2	2	3	4	4	5	5
68	2608	2610	2612	2613	2615	2617	2619	2621	2623	2625	0	0	1	1	1	1	1	2	2
	8246	8252	8258	8264	8270	8276	8283	8289	8295	8301	1	1	2	2	3	4	4	5	5
69	2627	2629	2631	2632	2634	2636	2638	2640	2642	2644	0	0	1	1	1	1	1	2	2
	8307	8313	8319	8325	8331	8337	8343	8349	8355	8361	1	1	2	2	3	4	4	5	5
70	2646	2648	2650	2651	2653	2655	2657	2659	2661	2663	0	0	1	1	1	1	1	2	2
	8367	8373	8379	8385	8390	8396	8402	8408	8414	8420	1	1	2	2	3	4	4	5	5
71	2665	2666	2668	2670	2672	2674	2676	2678	2680	2681	0	0	1	1	1	1	1	1	2
	8426	8432	8438	8444	8450	8456	8462	8468	8473	8479	1	1	2	2	3	3	4	5	5
72	2683	2685	2687	2689	2691	2693	2694	2696	2698	2700	0	0	1	1	1	1	1	1	2
	8485	8491	8497	8503	8509	8515	8521	8526	8532	8538	1	1	2	2	3	3	4	5	5
73	2702	2704	2706	2707	2709	2711	2713	2715	2717	2718	0	0	1	1	1	1	1	1	2
	8544	8550	8556	8562	8567	8573	8579	8585	8591	8597	1	1	2	2	3	3	4	5	5
74	2720	2722	2724	2726	2728	2729	2731	2733	2735	2737	0	0	1	1	1	1	1	1	2
	8602	8608	8614	8620	8626	8631	8637	8643	8649	8654	1	1	2	2	3	3	4	5	5
75	2739	2740	2742	2744	2746	2748	2750	2751	2753	2755	0	0	1	1	1	1	1	1	2
	8660	8666	8672	8678	8683	8689	8695	8701	8706	8712	1	1	2	2	3	3	4	5	5
76	2757	2759	2760	2762	2764	2766	2768	2769	2771	2773	0	0	1	1	1	1	1	1	2
	8718	8724	8729	8735	8741	8746	8752	8758	8764	8769	1	1	2	2	3	3	4	5	5
77	2775	2777	2778	2780	2782	2784	2786	2787	2789	2791	0	0	1	1	1	1	1	1	2
	8775	8781	8786	8792	8798	8803	8809	8815	8820	8826	1	1	2	2	3	3	4	4	5

The first significant figure and the position of the decimal point must be determined by inspection.

SQUARE ROOTS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
78	2793	2795	2796	2798	2800	2802	2804	2805	2807	2809	0	0	1	1	1	1	1	1	2
	8832	8837	8843	8849	8854	8860	8866	8871	8877	8883	1	1	2	2	3	3	4	4	5
79	2811	2812	2814	2816	2818	2820	2821	2823	2825	2827	0	0	1	1	1	1	1	1	2
	8888	8894	8899	8905	8911	8916	8922	8927	8933	8939	1	1	2	2	3	3	4	4	5
80	2828	2830	2832	2834	[illegible]	2837	2839	2841	2843	2844	0	0	1	1	1	1	1	1	2
	8944	8950	8955	8961	8967	8972	8978	8983	8989	8994	1	1	[illegible]	2	3	3	4	4	5
81	2846	2848	2850	2851	2853	2855	2857	2858	2860	2862	0	0	1	1	1	1	1	1	2
	9000	9006	9011	9017	9022	[illegible]	9033	9039	9044	9050	1	1	2	2	3	3	4	4	5
82	2864	2865	2867	2869	2871	2872	2874	2876	2877	2879	0	0	1	1	1	1	1		
	9055	9061	9066	9072	9077	9083	9088	9094	9099	9105	1	1	2	2	3	3			
83	2881	2883	2884	2886	2888	2890	2891	2893	2895	2897	0	0	1	1	1	1	1	[illegible]	2
	9110	9116	9121	9127	9132	9138	9143	9149	9154	9160	1	1	2	2	3	3	4	[illegible]	5
84	2898	2900	2902	2903	2905	2907	2909	2910	2912	2914	0	0	1	1	1	1	1	1	[illegible]
	9165	9171	9176	9182	9187	9192	9198	9203	9209	9214	1	1	2	2	3	3	4	4	5
85	2915	2917	2919	2921	2922	2924	2926	2927	2929	2931	0	0	1	1	1	1	1	1	2
	9220	9225	9230	9236	9241	9247	9252	9257	9263	9268	1	1	2	2	3	3	4	4	5
86	2933	2934	2936	2938	2939	2941	2943	[illegible]	2946	2948	0	0	1	1	[illegible]	1	1	1	2
	9274	9279	9284	9290	9295	9301	9306	9311	9317	9322	1	1	2	2	3	3	4	4	5
87	2950	2951	2953	2955	2956	2958	2960	2961	2963	2965	0	[illegible]	[illegible]	1	1	1	1	1	2
	9327	[illegible]	9338	9343	9349	9354	9359	9365	9370	9375	1	1	2	2	3	3	4	4	5
88	2966	2968	2970	2972	2973	2975	2977	2978	2980	2982	0	0	[illegible]	1	1	1	1	1	2
	9381	9386	9391	9397	9402	9407	9413	9418	9423	9429	1	1	2	2	3	3	4	4	5
89	2983	2985	2987	2988	2990	2992	2993	2995	2997	2998	0	0	1	1	1	1	1	1	2
	9434	9439	9445	9450	9455	9460	9466	9471	9476	9482	1	1	2	2	3	3	4	4	[illegible]
90	3000	3002	3003	3005	3007	3008	3010	3012	3013	[illegible]	[illegible]	0	0	1	1	1	1	1	1
	9487	9492	9497	9503	9508	9513	9518	9524	9529	9534	1	1	2	2	3	3	4	4	5
91	3017	3018	3020	3022	3023	3025	3027	[illegible]	3030	3032	0	0	0	1	1	1	1	1	1
	9539	9545	9550	9555	9560	9566	9571	9576	9581	9586	1	1	2	2	3	3	4	4	5
92	3033	3035	3036	3038	3040	3041	3043	3045	3046	3048	0	0	0	1	1	1	1	1	1
	9592	9597	9602	9607	9612	9618	9623	9628	9633	9638	1	1	2	2	3	3	4	4	5
93	3050	3051	3053	3055	3056	3058	3059	3061	3063	3064	0	0	0	1	1	1	1	1	1
	9644	9649	9654	9659	9664	9670	9675	9680	9685	9690	1	1	2	2	3	3	4	4	5
94	3066	3068	3069	3071	3072	3074	3076	3077	3079	3081	0	0	0	1	1	1	1	1	1
	9695	9701	9706	9711	9716	9721	9726	9731	9737	9742	1	1	2	2	3	3	4	4	5
95	3082	3084	3085	3087	3089	3090	3092	3094	3095	3097	0	0	0	1	1	1	1	1	1
	9747	9752	9757	9762	9767	9772	9778	9783	9788	9793	1	1	2	2	3	3	4	4	5
96	3098	3100	3102	3103	3105	3106	3108	3110	3111	3113	0	0	0	1	1	1	1	1	1
	9798	9803	9808	9813	9818	9823	9829	9834	9839	9844	1	1	2	2	3	3	4	4	5
97	3114	3116	3118	3119	3121	3122	3124	3126	3127	3129	0	0	0	1	1	1	1	1	1
	9849	9854	9859	9864	9869	[illegible]	[illegible]	9884	9889	9894	1	1	2	2	3	3	4	4	5
98	3130	3132	3134	3135	3137	3138	3140	3142	3143	3145	0	0	0	[illegible]	1	1	1	1	1
	9899	9905	9910	9915	9920	9925	9930	9935	9940	9945	0	1	1	2	2	3	3	4	4
99	3146	3148	3150	3151	3153	3154	3156	3158	3159	3161	0	0	0	[illegible]	1	1	1	1	1
	9950	9955	9960	9965	9970	9975	9980	9985	9990	9995	0	1	1	2	2	3	3	4	4

The first significant figure and the position of the decimal point must e determined by inspection.